MA8353 TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS L T P C 4 0 0 4

OBJECTIVES :

To introduce the basic concepts of PDE for solving standard partial differential equations.
To introduce Fourier series analysis which is central to many applications in engineering apart from its use in solving boundary value problems.
To acquaint the student with Fourier series techniques in solving heat flow problems used in various situations.
To acquaint the student with Fourier transform techniques used in wide variety of situations.
To introduce the effective mathematical tools for the solutions of partial differential equations that model several physical processes and to develop Z transform techniques for discrete time systems.

UNIT I PARTIAL DIFFERENTIAL EQUATIONS 12

Formation of partial differential equations – Singular integrals - Solutions of standard types of first order partial differential equations - Lagrange's linear equation - Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

UNIT II FOURIER SERIES 12

Dirichlet's conditions – General Fourier series – Odd and even functions – Half range sine series –Half range cosine series – Complex form of Fourier series – Parseval's identity – Harmonic analysis.

UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS 12

Classification of PDE – Method of separation of variables - Fourier Series Solutions of one dimensional wave equation – One dimensional equation of heat conduction – Steady state solution of two dimensional equation of heat conduction.

UNIT IV FOURIER TRANSFORMS 12

Statement of Fourier integral theorem – Fourier transform pair – Fourier sine and cosine transforms – Properties – Transforms of simple functions – Convolution theorem – Parseval's identity.

UNIT V Z - TRANSFORMS AND DIFFERENCE EQUATIONS 12

Z-transforms - Elementary properties – Inverse Z-transform (using partial fraction and residues) – Initial and final value theorems - Convolution theorem - Formation of difference equations – Solutionof difference equations using Z - transform.

TOTAL : 60 PERIODS

OUTCOMES :

Upon successful completion of the course, students should be able to:
Understand how to solve the given standard partial differential equations.
Solve differential equations using Fourier series analysis which plays a vital role in engineering applications.
Appreciate the physical significance of Fourier series techniques in solving one and two dimensional heat flow problems and one dimensional wave equations.

Understand the mathematical principles on transforms and partial differential equations would provide them the ability to formulate and solve some of the physical problems of engineering. Use the effective mathematical tools for the solutions of partial differential equations by using Z transform techniques for discrete time systems.

TEXT BOOKS :

1. Grewal B.S., “Higher Engineering Mathematics", 43rd Edition, Khanna Publishers, New Delhi,2014.

2. Narayanan S., Manicavachagom Pillay.T.K and Ramanaiah.G "Advanced Mathematics for Engineering Students", Vol. II & III, S.Viswanathan Publishers Pvt. Ltd, Chennai, 1998.

REFERENCES :

1. Andrews, L.C and Shivamoggi, B, "Integral Transforms for Engineers" SPIE Press, 1999.

2. Bali. N.P and Manish Goyal, "A Textbook of Engineering Mathematics", 9th Edition, Laxmi Publications Pvt. Ltd, 2014.

3. Erwin Kreyszig, "Advanced Engineering Mathematics ", 10th Edition, John Wiley, India, 2016.

4. James, G., "Advanced Modern Engineering Mathematics", 3rd Edition, Pearson Education, 2007.

5. Ramana. B.V., "Higher Engineering Mathematics", McGraw Hill Education Pvt. Ltd, New Delhi,2016.

6. Wylie, R.C. and Barrett, L.C., “Advanced Engineering Mathematics “Tata McGraw Hill Education Pvt. Ltd, 6th Edition, New Delhi, 2012.

MAILAM ENGINEERING COLLEGE

MAILAM (PO), Villupuram (DT). Pin: 604 304
(Approved by AICTE New Delhi, Affiliated to Anna University- Chennai
A TATA Consultancy Services Accredited Institution)

II Year B.E (Civil, EEE & Mech.)

DEPARTMENT OF MATHEMATICS

SUBJECT NAME : MA8353 - TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS

UNIT - I (PARTIAL DIFFERENTIAL EQUATIONS)

Syllabus:

Formation of partial differential equations – Singular integrals -- Solutions of standard types of first order partial differential equations - Lagrange's linear equation -- Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

No. of pages : 73+1+1+1=76 Cost per set: Rs.

Updated Questions:

Year	Nov-2017		April-2018	
	Q. No.	Pg. No.	Q. No.	Pg. No.
Part - A	14	5	22	8
	21	7	32	11
Part - B	52	66	49	63
	11	27	1	19
	33	49	27	42
	20	36	19	35

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PRINCIPAL

UNIT -I
PARTIAL DIFFERENTIAL EQUATIONS
PART – A (2-Marks Questions)

1. Explain the formulation of PDE(Partial differential equation) by elimination of arbitrary conditions.

Solution:

Let $f(x, y, z, a, b) = 0$ …. (1) Be an equation which conditions two arbitrary Constants 'a' and 'b'. Partially differentiating (1) with respect to (w .r .to) x and y We get two more equations using these three equations we can eliminate the two arbitrary constants a and b and finally we get the required PDE

2. Form a PDE by eliminating the arbitrary constants from z= ax + by

Solution:

Given $z = ax + by$ ….. (1)

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \quad \ldots\ldots (2)$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \quad \ldots\ldots (3)$$

Substituting (2) & (3) in (1), we get $z = px + qy$ which is the required PDE

3. Eliminating the arbitrary constants a and b from $z = ax + by + ab$

Solution:

Given $z = ax + by + ab$ ….. (1)

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \quad \ldots\ldots \mathbf{(2)}$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \quad \ldots\ldots \mathbf{(3)}$$

Substituting (2) in (1) & (3) in (1), we get $z = px + qy + pq$ **,** which is the required PDE

4. Form a PDE by eliminating the arbitrary constants from $z = (x+a)^2 + (y-b)^2$

Solution: **[AU M/J 2009, 2007]**

Given: $z = (x+a)^2 + (y-b)^2$ ….. (1)

$$p = \frac{\partial z}{\partial x} = 2(x+a) \Rightarrow x + a = \frac{p}{2} \ldots\ldots (2)$$

$$q = \frac{\partial z}{\partial y} = 2(y-b) \Rightarrow y - b = \frac{q}{2} \ldots\ldots (3)$$

Substituting (2)&(3) in(1), we get $z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$, $4z = p^2 + q^2$, which is the required PDE

5. Form a PDE by eliminating a and b form $z = (x^2 + a^2)(y^2 + b^2)$

Solution: **[AU M/J 2007 N/D 2009]**

Given: $z = (x^2 + a^2)(y^2 + b^2)$ ….. (1)

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b^2) \Rightarrow y^2 + b^2 = \frac{p}{2x} \quad \ldots\ldots (2)$$

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a^2) \quad \Rightarrow x^2 + a^2 = \frac{q}{2y} \qquad \text{..... (3)}$$

Substituting (2)&(3) in (1), we get $z = \frac{q}{2y} \cdot \frac{p}{2x}$

$pq = 4xyz$, which is the required PDE

6. Form a PDE by eliminating the arbitrary constants in $z = (x-a)^2 + (y-b)^2 + 1$

Solution: **[AU M/J 2011 N/D 2007,2008]**

Given:

$$z = (x-a)^2 + (y-b)^2 + 1 \quad \text{..... (1)}$$

$$p = \frac{\partial z}{\partial x} = 2(x-a) \Rightarrow x - a = \frac{p}{2} \text{.....(2)}$$

$$q = \frac{\partial z}{\partial y} = 2(\text{y-b}) \quad \Rightarrow \text{y-b} = \frac{\text{q}}{2} \text{.....(3)}$$

Substituting (2)&(3) in(1), we get $z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 + 1$

$4z = p^2 + q^2 + 4$, which is the required PDE

7. Form a partial differential equation by eliminating the constants a **and** b **from** $z = ax^n + by^n$

Solution: **[AU M/J 2008, 2009]**

Given:

$$z = ax^n + by^n \quad \text{..... (1)}$$

$$p = \frac{\partial z}{\partial x} = anx^{n-1} \Rightarrow a = \frac{p}{nx^{n-1}} \qquad \text{...... (2)}$$

$$q = \frac{\partial z}{\partial y} = bny^{n-1} \Rightarrow b = \frac{q}{ny^{n-1}} \qquad \text{..... (3)}$$

Substituting (2) & (3) in (1), we get

$$z = \frac{p}{nx^{n-1}} x^n + \frac{q}{ny^{n-1}} y^n$$

$$z = \frac{1}{n}(px + qy)$$

$px + qy = nz$, which is the required PDE

8. Form a partial differential equation by eliminating the constants a **and** b **form** $z = ax^2 + by^2$

Solution: **[AU N/D 2013]**

Given: $z = ax^2 + by^2 \qquad \text{..... (1)}$

$$p = \frac{\partial z}{\partial x} = 2ax \Rightarrow a = \frac{p}{2x} \qquad \text{...... (2)}$$

$$q = \frac{\partial z}{\partial y} = 2by \Rightarrow b = \frac{q}{2y} \qquad \text{..... (3)}$$

Substituting (2) & (3) in (1), we get

$$z = \frac{p}{2x}x^2 + \frac{q}{2y}y^2$$

$$z = \frac{1}{2}(px + qy)$$

$px + qy = 2z$, which is the required PDE

9. Form a partial differential equation by eliminating the constants a and b from $z = ax^3 + by^3$

Solution: **[AU M/J 2014]**

Given: $z = ax^3 + by^3$ ….. (1)

$$p = \frac{\partial z}{\partial x} = 3ax^2 \Rightarrow a = \frac{p}{3x^2} \quad \text{…… (2)}$$

$$q = \frac{\partial z}{\partial y} = 3by^2 \Rightarrow b = \frac{q}{3y^2} \quad \text{….. (3)}$$

Substituting (2) & (3) in (1), we get

$$z = \frac{p}{3x^2}x^3 + \frac{q}{3y^2}y^3$$

$$z = \frac{1}{3}(px + qy)$$

$px + qy = 3z$, which is the required PDE

10. Form a PDE by eliminating the arbitrary constants from $z = (2x^2 + a)(3y - b)$

Solution: **[AU A/M 2008,2009]**

Given $z = (2x^2 + a)(3y - b)$ ….. (1)

$$p = \frac{\partial z}{\partial x} = 4x(3y - b) \Rightarrow \frac{p}{4x} = 3y - b \quad \text{…… (2)}$$

$$q = \frac{\partial z}{\partial y} = (2x^2 + a)\cdot 3 \Rightarrow \frac{q}{3} = 2x^2 + a \quad \text{….. (3)}$$

Substituting (2) & (3) in (1), we get $pq = 12xz$, which is the required PDE

11. Form a PDE by eliminating the arbitrary constants from $(x-a)^2 + (y-b)^2 + z^2 = 1$

Solution: **[AU M/J 2011, N/D 2007,2008]**

Given $(x-a)^2 + (y-b)^2 + z^2 = 1$ ….. (1)

Partially differentiating with respect to (P.d.w.r.to) ‘x’ & ‘y’ we get

$2(x-a) + 2zp = 0 \Rightarrow x - a = -zp$ …… (2)

$2(y-b) + 2zq = 0 \Rightarrow y - b = -zq$ ….. (3)

Substituting (2) & (3) in (1), we get

$(-zp)^2 + (-zq)^2 + z^2 = 1$

$\therefore z^2(p^2 + q^2 + 1) = 1$, which is the required PDE

12. Find the PDE of the family of sphere having their center on the line $x = y = z$

Solution: **[AU N/D 2004]**

The equation of the sphere is $(x-a)^2+(y-a)^2+(z-a)^2=r^2$ ….. (1)

P.d.w.r.to 'x' & 'y' we get

$2(x-a)+2(z-a)p=0$

$x-a+(z-a)p=0$

$(x+zp)-a(1+p)=0$

$a(1+p)=x+zp \Rightarrow a=\dfrac{x+zp}{1+p}$ …… (2)

$2(y-a)+2(z-a)q=0$

$y-a+(z-a)q=0$

$(y+zq)-a(1+q)=0$

$a(1+q)=y+zq \Rightarrow a=\dfrac{y+zq}{1+q}$ ….. (3)

From (2) & (3) $\Rightarrow \dfrac{x+zp}{1+p}=\dfrac{y+zq}{1+q}$

$(x+zp)(1+q)=(1+p)(y+zq)$

$x+qx+zp+zpq=y+zq+py+pqz$

$\therefore p(z-y)+q(x-z)=y-x$, which is the required PDE.

13. Obtain the PDE by eliminating the arbitrary constant a and b from

$z=xy+y\sqrt{x^2-a^2}+b$

Solution: **[AU N/D 2011, A/M 2007]**

Given: $z=xy+y(x^2-a^2)^{\frac{1}{2}}+b$ ….. (1)

$p=\dfrac{\partial z}{\partial x}=y+y\dfrac{1}{2}(x^2-a^2)^{-\frac{1}{2}}(2x)$ …… (2)

$q=\dfrac{\partial z}{\partial y}=x+(x^2-a^2)^{\frac{1}{2}}$ ….. (3)

(3)$\Rightarrow$ $(x^2-a^2)^{\frac{1}{2}}=q-x$

$(x^2-a^2)=(q-x)^2$ ….. (4)

Substitute (4) in (2), we have

$p=y+xy\left[(q-x)^2\right]^{-\frac{1}{2}}$

$=y+xy(q-x)^{-1}$

$=y+\dfrac{xy}{q-x}=\dfrac{y(q-x)+xy}{q-x}$

$p(q-x)=yq-xy+xy$

$p(q-x)=qy$

14. Find the partial differential equation by eliminating the arbitrary function f from the

relation $z = f(x^2 - y^2)$

Solution: **[AU A/M 2008, N/D 2017]**

Given:

$$z = f(x^2 - y^2) \quad \ldots\ldots (1)$$

d.p.w.r.to 'x' & 'y' we get ,

$$p = \frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x)$$

$$\Rightarrow f'(x^2 - y^2) = \frac{p}{2x} \quad \ldots\ldots (2)$$

$$q = \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y)$$

$$\Rightarrow f'(x^2 - y^2) = \frac{-q}{2y} \quad \ldots\ldots (3)$$

From (2) & (3), we have

$$\frac{p}{2x} = \frac{-q}{2y} \Rightarrow py = -qx$$

$py + qx = 0$, which is the required PDE

15. Form a PDE by eliminating the function from the relation $z = f\left(\frac{y}{x}\right)$

Solution: **[AU N/D 2005, 2014]**

Given: $z = f\left(\frac{y}{x}\right) \quad \ldots\ldots (1)$

P.d.w.r.to 'x' & 'y' we get,

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow f'\left(\frac{y}{x}\right) = \frac{-px^2}{y} \quad \ldots\ldots (2)$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$\Rightarrow f'\left(\frac{y}{x}\right) = qx \quad \ldots\ldots (3)$$

From (2) & (3) we get,

$$\frac{-px^2}{y} = qx \Rightarrow -px^2 = qxy$$

$px + qy = 0$, which is the required PDE

16. Form a PDE by eliminating the arbitrary function f **from the relation**

$$z = x^2 + 2g\left(\frac{1}{y} + \log x\right)$$

Solution: **[AU N/D 2011, A/M 2007]**

Given: $z = x^2 + 2g\left(\frac{1}{y} + \log x\right)$ ….. (1)

P.d.w.r.to 'x' & 'y' we get,

$$p = \frac{\partial z}{\partial x} = 2x + 2g'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right) \qquad \text{…… (2)}$$

$$q = \frac{\partial z}{\partial y} = 2g'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2}\right)$$

$$2g'\left(\frac{1}{y} + \log x\right) = -qy^2 \qquad \text{….. (3)}$$

Substitute (3) in (2), we have

$$p = 2x - \frac{qy^2}{x} = \frac{2x^2 - qy^2}{x} \qquad px - 2x^2 = -qy^2$$

$$px + qy^2 = 2x^2$$

17. Eliminating the arbitrary function f from $z = f\left(\frac{xy}{z}\right)$ and form the PDE.

Solution: **[AU A/M 2004, 2007]**

Given $z = f\left(\frac{xy}{z}\right)$

P.d.w.r.to x &y we get,

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left(\frac{zy - xyp}{z^2}\right) \qquad \text{….. (1)}$$

$$f'\left(\frac{xy}{z}\right) = \frac{pz^2}{zy - xyp} \qquad \text{…… (2)}$$

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left(\frac{zx - xyq}{z^2}\right)$$

$$f'\left(\frac{xy}{z}\right) = \frac{qz^2}{zx - xyq} \qquad \text{…… (3)}$$

From (2) & (3), we have

$$\frac{pz^2}{zy - xyp} = \frac{qz^2}{zx - xyq}$$

$$qy(z - xp) = px(z - yq)$$

$$qyz - pqxy = pxz - pqxy$$

$px = qy$, which is the required PDE

18. Eliminating the arbitrary functions f and g from $z = f(x+iy) + g(x-iy)$ to obtain a partial differential equation involving x, y, z.

Solution:

Given $z = f(x+iy)+g(x-iy)$

$$p = \frac{\partial z}{\partial x} = f'(x+iy)+g'(x-iy)$$

$$q = \frac{\partial z}{\partial y} = if'(x+iy)-ig'(x-iy)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy)+g''(x-iy)$$

$$t = \frac{\partial^2 z}{\partial y^2} = -f''(x+iy)-g''(x-iy)$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

19. Solve $\frac{\partial z}{\partial x} = \sin x$ **[AU A/M 2007, 2008]**

Solution:

Given: $\frac{\partial z}{\partial x} = \sin x$

Integrating partially with respect to y

$$\int \frac{\partial z}{\partial x} = \int \sin x \partial x = \int \sin x dx$$

$z = -\cos x + c$, where $f(x)$ is arbitrary. function

20. Solve $\frac{\partial^2 z}{\partial x \partial y} = 0$

Solution:

Given: $\frac{\partial^2 z}{\partial x \partial y} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0$$

Integrating partially with respect to 'x'

$$\left(\frac{\partial z}{\partial y}\right) = f(y)$$

Again integrating $\int \partial z = \int f(y)\partial y$

$$z = F(y) + d(x)$$

$$\therefore F(y) = \int f(y)\partial y$$

21. Solve $\sqrt{p}+\sqrt{q}=1$ **[AU A/M 2008, N/D 2017]**

Solution:

$Given\sqrt{p}+\sqrt{q}=1$ ….. (A)

This is of the type $F(p,q) = 0$

$\therefore$ The trial solution is $z = ax + by + c$ ….. (1)

To find complete integral (C.I):

$$(1) \Rightarrow \left. \begin{aligned} p &= \frac{\partial z}{\partial x} = a \\ q &= \frac{\partial z}{\partial y} = b \end{aligned} \right\} \quad \text{….. (2)}$$

$\therefore \sqrt{a} + \sqrt{b} = 1$ [Using (2) in (A)]

$$\sqrt{b} = (1 - \sqrt{a})$$

$$b = (1 - \sqrt{a})^2 \quad \text{….. (3)}$$

Substitute (3) in (1) we get $z = ax + (1 - \sqrt{a})^2 y + c$ ….. (4)

Which is a Complete Integral.

To find singular Integral (S.I):

There is no S.I for this type

To find general Integral (G.I):

Let $c = f(a)$ in (4), we get

$$z = ax + (1 - \sqrt{a})^2 y + f(a) \quad \text{….. (5)}$$

$$\frac{\partial z}{\partial a} = 0 \Rightarrow x + 2(1 - \sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right) y + f'(a) = 0) \quad \text{….. (6)}$$

Eliminating 'a' from (5) & (6) we get the general solution of the given pde.

22. Find the complete integral and Solve $z = px + qy + 2\sqrt{pq}$ **[AU A/M 2018, N/D 2004]**

Solution:

Given $z = px + qy + 2\sqrt{pq}$ ….. (1)

This is of the type

$z = px + qy + f(p,q)$ (Clairaut's form)

The C.I is $z = ax + by + 2\sqrt{ab}$

To find singular Integral (S.I):

$$(1) \Rightarrow \frac{\partial z}{\partial a} = 0 \Rightarrow x + \sqrt{\frac{b}{a}} = 0$$

$$x = -\sqrt{\frac{b}{a}} \quad \text{….. (2)}$$

$$\frac{\partial z}{\partial b} = 0 \Rightarrow y + \sqrt{\frac{a}{b}} = 0$$

$$y = -\sqrt{\frac{a}{b}} \quad \text{….. (3)}$$

Eliminating 'a' & 'b' from (2) & (3) we get

$xy = 1$, which is singular integral

23. Solve $z = px + qy + (pq)^{3/2}$ (OR) *Find the Complete integral of* $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$

[AU N/D 2005, 2009 , N/D 2016]

Solution: Multiply both sides by pq

$$Given \;\; z = px + qy + (pq)^{3/2}$$

Complete integral is $z = ax + by + (ab)^{\frac{3}{2}}$

24. Find the complete integral of $z = px + qy + p^2 + q^2$

Solution: **[AU N/D 2008]**

$$Given \;\; z = px + qy + p^2 + q^2$$

This is the type $z = px + qy + f(p,q)$

The C.I is $z = ax + by + a^2 + b^2$

25. Solve $p + q = x - y$ **[AU N/D 2007, A/M 2008]**

Solution:

$$Given \;\; p + q = x - y$$

$$x - p = q + y = k$$

$$x - p = k, q + y = k$$

$$p = x - k, q = k - y$$

$$z = \int pdx + \int qdy$$

$$= \int (x-k)dx + \int (k-y)dy$$

$$= \frac{(x-k)^2}{2} + \frac{(k-y)^2}{2}$$

$$2z = (x-k)^2 + (k-y)^2$$

26. Solve the equation $yp = 2yx + \log q$ **[AU N/D 2001, A/M 2005]**

Solution:

$$Given \;\; yp = 2yx + \log q$$

$$F_1(x,p) = F_2(y,q) = k$$

$$y(p - 2x) = \log q$$

$$p - 2x = \frac{\log q}{y} = k$$

$p - 2x = k$	$\log q - ky = 0$
$p = 2x + k$	$\log q = ky$

$$q = e^{ky}$$

$$z = \int pdx + \int qdy$$

$$= \int (2x + k)dx + \int e^{ky} dy$$

$$z=(x^2+kx)+\frac{e^{ky}}{k}+c$$

This is C.I. there is no S.I

27. Solve the equation $p\tan x+q\tan y=\tan z$

Solution:

Given: $p\tan x+q\tan y=\tan z$

Lagrange's type $Pp+Qq=R$

Here $p\to \tan x, Q\to \tan y, R\to \tan z$

The S.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$$

Take

$$\frac{dx}{\tan x}=\frac{dy}{\tan y}$$

$$\cot x dx=\cot y dy$$

Integrating on both sides

$$\log(\sin x)=\log(\sin y)+\log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right)=\log c_1$$

$$u=\frac{\sin x}{\sin y}c_1$$

$$\frac{dy}{\tan y}=\frac{dz}{\tan z}$$

$$\cot y dy=\cot z dz$$

$$\log(\sin y)=\log(\sin z)+\log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right)=\log c_2$$

$$v=\frac{\sin y}{\sin z}c_2$$

The solution of the given PDE is

$$\phi\left(\frac{\sin x}{\sin y},\frac{\sin y}{\sin z}\right)=0$$

28. Find the solution of $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$ **[AU A/M 2006, 2007,2010]**

Solution:

Given $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

Lagrange's type $Pp+Qq=R$ here $p\to \tan x, Q\to \tan y, R\to \tan z$

The S.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$$

Taking

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} \qquad\qquad \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}} \qquad\qquad \int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$2\sqrt{x} = 2\sqrt{y} + 2c_1 \qquad\qquad 2\sqrt{y} = 2\sqrt{z} + 2c_2$$

$$\sqrt{x} = \sqrt{y} + c_1 \qquad\qquad \sqrt{y} = z + c_2$$

$$\sqrt{x} - \sqrt{y} = c_1 \qquad\qquad \sqrt{y} - z = c_2$$

$$u = \sqrt{x} - \sqrt{y} \qquad\qquad v = \sqrt{y} - \sqrt{z}$$

The solution of the given PDE is $\phi\left(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}\right) = 0$

29. Write the subsidiary equation for $x^2 p + y^2 q = (x+y)z$

Solution:

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

30. Find the general solution of $4\frac{\partial^2 z}{\partial y^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial^2 y} = 0$

Solution:

The auxiliary equation is $4m^2 - 12m + 9 = 0$

$$m = \frac{12 + \sqrt{144 - 144}}{8}$$

$$m = 3/2, 3/2 \Rightarrow \text{Equal roots}$$

$$\therefore z = f_1\left[y + \left(\frac{3}{2}x\right)\right] + xf_2\left[y + \left(\frac{3}{2}x\right)\right]$$

31. Find the solution of $4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ **[AU A/M 2005, 2006]**

Solution:

The auxiliary equation is $4m^2 - 1 = 0$

$$m^2 = 1/4 \Rightarrow m = \pm 1/2$$

$$\therefore z = f_1\left[y + \left(\frac{1}{2}x\right)\right] + f_2\left[y - \left(\frac{1}{2}x\right)\right]$$

32. Solve the equation $(D^3 - 3DD'^2 + 2D'^3)z = 0$ **[AU A/M 2018]**

Solution:

The auxiliary equation is $m^3 - 3m + 2 = 0$

$$(m-1)\left(m^2 + m - 2\right) = 0$$

$$(m-1)(m-1)(m+2)=0$$
$$m=1,1,-2$$
$$\therefore z= f_1[y+x]+xf_2[y+x]+f_3[y-2x]$$

33. Solve $(D^3-4D^2D^{'}+4DD^{'2})z=0$ **where** $D=\frac{\partial}{\partial x^{'}},D^{'}=\frac{\partial}{\partial y}$

Solution:

The auxiliary equation is $m^3-4m^2+4m=0$

$$m(m^2-4m+4)=0$$
$$m=0,2,2$$
$$\therefore z= f_1[y]+f_2[y+2x]+xf_3[y+2x]$$

34. Find the P.I of $(D^2+4DD^{'})z=e^x$

Solution:

$$P.I=\frac{1}{D^2+4DD^{'}}e^x$$
$$=\frac{1}{D^2+4DD^{'}}e^{x+0y}$$
$$=\frac{1}{1+4(1)(0)}e^{x+0y}$$

$$\left\{\begin{array}{l} a=1,b=0 \\ \text{Replace } D\to 1,D^{'}\to 0\end{array}\right.$$

P.I $=e^x$

35. Find the P.I of $(D^2+DD^{'})z=e^{x-y}$

Solution:

To find P.I:

$$P.I=\frac{1}{(D+D')^2}e^{x-y}$$
$$=\frac{1}{(1-1)^2}e^{x-y}$$
$$=\frac{x}{2(D+D^{'})}e^{x-y}$$
$$P.I=\frac{x^2}{2}e^{x-y}\qquad ,D\to 1,D^{'}\to -1$$

36. Find C.F of $(D^2+2DD^{'})z=0$

Solution:

The auxiliary equation is $m^2+2m=0$

$$m(m+2)=0$$
$$m=0,m=-2$$

C.F $=f_1(y+0x)+f_2(y-2x)$

37. What is the C.F of $(D^2-4DD^{'}+4D^2)z=x+y$

Solution:

The auxiliary equation is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$\text{C.F} = f_1(y+2x) + xf_2(y+2x)$$

38. Write the particular integral of $(D^2 - DD')z = \sin(x+y)$

Solution:

$$P.I = \frac{1}{D^2 - DD'}\sin(x+y) \qquad replace\, D^2 by -1, DD' by\ 1, D'^2\ by -1$$

$$= \frac{1}{-1+1}\sin(x+y) \Rightarrow \frac{x}{(2D-D')}\sin(x+y)$$

$$= \frac{-x}{2}\cos(x+y)$$

39. **Find the PDE of the family of spheres having their centers on the z-axis.**

Solution: **[AU N/D 2015]**

Let the centre of the sphere be (0,0,c) a point on the z-axis and its radius.
Its equations is

$$(x-0)^2 + (y-0)^2 + (z-0)^2 = k^2$$

$$ie., \qquad x^2 + y^2 + (z-c)^2 = k^2 \text{.............(1)}$$

Here c and k are arbitrary constants
Differentiating (1) w.r.to x, we get

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$x + p(z-c) = 0 \qquad \text{...... (2)}$$

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$2y + 2q(z-c) = 0$$

$$y + q(z-c) = 0 \qquad \text{.... (3)}$$

Eliminate 'c' from (2) & (3), we get

$$(2) \Rightarrow \quad x + p(z-c) = 0$$

$$z - c = \frac{-x}{p}$$

$$(3) \Rightarrow z - c = \frac{-y}{q}$$

$$\frac{-x}{p} = \frac{-y}{q}$$

$$qx = py$$

40. Solve the equation $(D - D')^3 z = 0$

Solution:

The auxiliary equation is

$$(m-1)^3 = 0$$

$$m = 1,1,1$$

$$\therefore z = \phi_1(y+x) + x\phi_2(y+x) + x^2\phi_3(y+x)$$

41. Form the PDE by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$

Solution:

The given equation is $z^2 - xy = f\left(\frac{x}{z}\right)$ ….. (1)

Diff (1) p.w.r.to x, we get

$$2z\frac{\partial z}{\partial x} - y = f'\left(\frac{x}{z}\right)\left[\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right]$$

$$2zp - y = f'\left(\frac{x}{z}\right)\left[\frac{z - xp}{z^2}\right] \qquad \text{….. (2)}$$

Diff (1) p.w.r.to y, we get

$$2z\frac{\partial z}{\partial y} - x = f'\left(\frac{x}{z}\right)\left[\frac{z(0) - x\frac{\partial z}{\partial y}}{z^2}\right]$$

$$2zq - x = f'\left(\frac{x}{z}\right)\left[\frac{-xq}{z^2}\right] \qquad \text{…. (3)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{2zp - y}{2zq - x} = \frac{z - xp}{-xq}$$

$$-2xzpq + xyq = 2z^2q - 2xzpq - xz + x^2p$$

$$xyq = 2z^2q - xz + x^2p$$

$$x^2p + 2z^2q - xyq = xz$$

$$x^2p - (xy - 2z^2)q = xz$$

42. Find the complete integral of $p = 2qx$

Solution:

Given $p = 2qx$, this equation is of the form $f(x, p, q) = 0$

Let $q = a$

Then $p = 2ax$

Put $dz = pdx + qdy$

$$dz = 2ax\,dx + ady$$

Integrating on both sides we get,

$$z = ax^2 + ay + c \qquad \text{…. (1)}$$

Equation (1) is the complete integral of the given equation.

Since the number of a.c =number of I.V
Differentiating p.w.r.to c, we get $1=0$
Hence there is no singular integral.
General integral can be found out in the usual way.

43. Solve $(D-1)(D-D'+1)z=0$ **[AU N/D 2012]**

Solution:

Given $(D-1)(D-D'+1)z=0$

$$[D-0D'-1][D-D'-(-1)]z=0$$

by working rule,

If $(D-mD-c)z=0, then\ \ z=e^{cx}f(y+mx), where\ c\ is\ abitrary$ costants

$$here, \quad m_1=0, \quad c_1=1$$
$$m_2=1, \quad c_2=-1$$
$$\therefore z=e^{x}f_1(y+0x)+e^{-x}f_2(y+x)$$
$$z=e^{x}f_1(y)+e^{-x}f_2(y+x)$$

44. Solve $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}=0$ **[AU N/D 2013]**

Solution:

Given $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}=0$

$$D^2-DD'+D=0$$
$$D(D-D'+1)=0$$

The roots are m = 0, 1 and c = 0, -1

$$z=e^{0x}f_1(y+0x)+e^{-x}f_2(y+x)$$
$$z=f_1(y)+e^{-x}f_2(y+x)$$

45. Solve $(D^4-D'^4)z=0$ **[AU M/J 2014]**

Solution:

Given: $(D^4-D'^4)z=0$

The auxiliary equation is $m^4-1=0$

$$(m^2+1)(m^2-1)=0$$
$$m=1,-1,i,-i$$
$$z=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix)$$

46. Find the complete integral of $q=2px$ **[AU M/J 2015]**

Solution:

Given $q=2px$, this equation is of the form

$$F_1(x,p)=F_2(x,p)$$

Let $q = 2px = k$

$q = k \qquad 2px = k$

$$p = \frac{k}{2x}$$

$$z = \int p\,dx + \int q\,dy$$

$$= \frac{k}{2}\int \frac{1}{x}dx + \int k\,dy$$

$$= \frac{k}{2}\log x + ky + b$$

Which is the complete integral of the given equation.
Hence there is no singular integral.

47. Form the partial differential equation by eliminating the arbitrary constants 'a' and 'b' from $\log(az-1) = x + ay + b$ **[AU A/M 2015]**

Sol:

$\log(az-1) = x + ay + b$...................(1)

Partially differentiating w.r.t 'x' and 'y' we get

$$\frac{1}{az-1}ap = 1$$

$$ap = (az-1)$$

$$a(z-p) = 1$$

$$\Rightarrow \qquad a = \frac{1}{z-p} \quad \text{.................(2)}$$

$$\frac{1}{az-1}aq = a$$

$$\Rightarrow \qquad q = az - 1 \quad \text{..............................(3)}$$

Sub. (2) in (3), we get

$$\Rightarrow \qquad q = \frac{z}{z-p} - 1$$

$$= \frac{z - z + p}{z-p}$$

$$q(z-p) = p$$

$$qz - pq = p$$

$$qz = p + pq$$

$$p(q+1) = zq$$

48. Find the complete solution of $p + q = 1$. **[AU N/D 2014]**

Solution:

$$p+q=1 \ldots\ldots\ldots\ldots\ldots(1)$$

This is of the type $F(p,q)=0$

The trial solution is $z=ax+by+c \ldots\ldots\ldots\ldots\ldots(2)$

$$\left.\begin{aligned} p&=\frac{\partial z}{\partial x}=a \\ q&=\frac{\partial z}{\partial y}=b \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(3)$$

Sub. (3) in (1), we get

$$a+b=1$$

$$b=1-a \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Using (4) in (2), we get

$$z=ax+(1-a)y+c \qquad \text{Which is C.I}$$

49. Form the partial differential equation by eliminating the arbitrary functions from $f(x^2+y^2, z-xy)=0$ **[AU M/J 2016]**

Solution:

Given $f(x^2+y^2, z-xy)=0$

$$u = x^2+y^2 \qquad v = z-xy$$

$$u_x = 2x \qquad v_x = p-y$$

$$u_y = 2y \qquad v_x = q-x$$

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x & 2y \\ p-y & q-x \end{vmatrix} = 0$$

$$2x(q-x)-2y(p-y)=0$$

$$2xq-2x^2-2yp+2y^2=0$$

$$2xq-2yp=2x^2-2y^2$$

$xq-yp=x^2-y^2$, which is the required PDE

50 Find the complete solution of the partial differential equation $p^3-q^3=0$

Solution: **[AU M/J 2016]**

Given:

$$p^3-q^3=0$$

$$f(p,q)=0$$

$$z=ax+by+c$$

Here $p = a, \quad q = b$

$$a^3 - b^3 = 0$$

$$a^3 = b^3 \qquad \Rightarrow a = b$$

$$\therefore z = ax + ay + c$$

51. Solve $(D + D' - 1)(D - 2D' + 3)z = 0$ **[AU N/D 2015]**

Solution:

Given

$$(D + D' - 1)(D - 2D' + 3)z = 0$$

$$[D - (-1)D' - 1][D - 2D' - (-3)]z = 0$$

by working rule,

If $(D - mD - c)z = 0$, *then* $z = e^{cx} f(y + mx)$, *where c is abitrary* costants

here, $m_1 = -1, \quad c_1 = 1$

$$m_2 = 1, \quad c_2 = -3$$

$$\therefore \quad z = e^x f_1(y - x) + e^{-3x} f_2(y + 2x)$$

52. **Form the partial differential equation by eliminating arbitrary function 'f' from** $z = e^{ay} f(x + by)$ **[AU A/M 2017]**

Solution: Given $z = e^{ay} f(x + by)$ --------(i)

Differentiate Partially (i) w.r.t x, and y

$$p = e^{ay} f'(x + by) - - - - (2)$$

$$q = e^{ay} af'(x + by)\ b - - - - (3)$$

Compare 2 and 3

$$q = ab\ e^{ay} f'(x + by) \qquad \Rightarrow q = ab\ p \qquad \Rightarrow \frac{q}{p} = ab$$

53. Solve $(D^3 - D^2 D' - 8DD'^2 + 12D'^3)z = 0$ **[AU A/M 2017]**

Solution: Given $(D^3 - D^2 D' - 8DD'^2 + 12D'^3)z = 0$

The auxillary eqn. is $m^3 - m^2 - 8m + 12 = 0$

The three roots are m = 2, 2, -3

$$C.F \therefore z = f_1[y + 2x] + xf_2[y + 2x] + f_3[y - 3x]$$

54. **Find the PDE of all spheres whose centre lie on the x-axis.** **[AU N/D 2016]**

Solution: The eqn. of the sphere whose centre lie on the x-axis is

$$(x - a)^2 + y^2 + z^2 = 1$$

Diff. partially w.r.to x and y is

$$2(x - a) + 2zp = 0$$

$$2y + 2zq = 0$$

$y + zq = 0$ Which is the required PDE

PART –B

PROBLEMS BASED ON LANRANGE'S LINEAR EQUATIONS

1. Solve $x(y-z)p + y(z-x)q = z(x-y)$ **[AU N/D 2011, 2014 A/M 2018]**

Solution:

Given $x(y-z)p + y(z-x)q = z(x-y)$

Lagrange's type $Pp + Qq = R$

The S.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Choosing 1, 1, 1 as a Lagrange's multipliers, we get each of ratio are equal to

$$\frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy} = \frac{dx+dy+dz}{0}$$

$$d(x+y+z) = 0$$

$$\int d(x+y+z) = 0$$

$$x+y+z = c_1$$

$$u = x+y+z$$

Choosing $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as Lagrange's Multipliers, we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2 ; xyz = c_2$$

$$v = xyz$$

Result: The solution of given PDE is $\phi(x+y+z, xyz) = 0$

2. Solve $(mz - ny)p + (nx - lz)q = ly - mx$ **[AU N/D 16 , A/M 16]**

Solution:

Given $(mz - ny)p + (nx - lz)q = ly - mx$

This is of the form Lagrange's type

$$Pp + Qq = R$$

The S.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{(mz - ny)} = \frac{dy}{(nx - lz)} = \frac{dz}{(ly - mx)}$$

Choosing l, m, n as Lagrange's multipliers we get

$$\frac{(ldx + m\,dy + ndz)}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$

$$ldx + mdy + ndz = 0$$

$$\int ldx + \int mdy + \int ndz = 0$$

$$lx + my + nz = C_1$$

$$lx + my + nz = u$$

Using Lagrange's multipliers x, y, z we get each of above ratio is equal to

$$\frac{(x\,dx + y\,dy + z\,dz)}{(x(mz - ny) + y(nx - lz) + z(ly - mx))} = \frac{xdx + ydy + zdz}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int xdx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2 \;;\; x^2 + y^2 + z^2 = 2C_2$$

$$v = x^2 + y^2 + z^2$$

Result: The solution of given PDE is $\phi(u, v) = 0$

$$\phi(lx + my + nz, x^2 + y^2 + z^2) = 0$$

3. Solve $(3z - 4y)\frac{\partial z}{\partial x} + (4x - 2z)\frac{\partial z}{\partial y} = 2y - 3x$ **[AU N/D 2008 ,M/J 2000]**

Solution:

Given $(3z - 4y)\frac{\partial z}{\partial x} + (4x - 2z)\frac{\partial z}{\partial y} = 2y - 3x$

Lagrange's type $Pp + Qq = R$

The S.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$$

Using x, y, z as Lagrange' s multiplier s, each of the above ratio

$$\frac{x\,dx + y\,dy + z\,dz}{3xz - 4xy + 4xy - 2yz + 2yz - 3xz} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int xdx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1}{2}\ ;x^2+y^2+z^2=c_1$$

$$u=x^2+y^2+z^2$$

Choose 2, 3, 4 as Lagrange' s multiplier each above ratio is equal to

$$\frac{2\,dx+3\,dy+4\,dz}{(6z-8y+12x-6z+8y-12x)}=\frac{2\,dx+3\,dy+4\,dz}{0}$$

$$2\,dx+3\,dy+4\,dz=0$$

$$2\int dx+3\int dy+4\int dz=0$$

$$2x+3y+4z=c_2\quad ;v=2x+3y+4z$$

Result: The Solution of given PDE is $\phi(u,v)=0$

$$\phi(x^2+y^2+z^2,2x+3y+4z)=0$$

4. Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$ **[AU N/D 2007,10,A/M 2002,2015, 2016]**

Solution:

Given $(x^2-yz)p+(y^2-zx)q=z^2-xy$

Lagrange' s type $Pp+Qq=R$

The S,E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{(x^2-yz)}=\frac{dy}{(y^2-zx)}=\frac{dz}{(z^2-xy)}\ldots\ldots\ldots\ldots\ldots(1)$$

Method of grouping is not possible

Using two set of multipliers x,y,z: 1,1,1 each to the ratio in (1)

$$=\frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz}$$

$$=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{xdx+y\,dy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{xdx+y\,dy+zdz}{(x+y+z)}=\frac{dx+dy+dz}{1}$$

$$x\,dx+y\,dy+z\,dz=(x+y+z).d(x+y+z)$$

$$\int x\,dx+\int y\,dy+\int z\,dz=\int (x+y+z).\ d(x+y+z)$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{(x+y+z)^2}{2}$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{x^2+y^2+z^2+2xy+2yz+2zx}{2}$$

$$xy+yz+zx=c_1$$

$$u=xy+yz+zx$$

Using two sets of multipliers 0,1,-1 each of the ratio in (1)

$$\frac{dx - dy}{(x^2 - yz) - (y^2 - zx)} = \frac{(dy - dz)}{(y^2 - zx) - (z^2 - xy)}$$

$$\frac{dx - dy}{(x^2 - y^2) + z(x - y)} = \frac{dy - dz}{(y^2 - z^2) + x(y - z)}$$

$$\frac{d(x - y)}{(x + y)(x - y) + z(x - y)} = \frac{d(y - z)}{(y + z)(y - z) + x(y - z)}$$

$$\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z))}{(y - z)(x + y + z)}$$

$$\int \frac{d(x - y)}{(x - y)} = \int \frac{d(y - z)}{(y - z)}$$

$$\log(x - y) = \log(y - z) + \log c_2$$

$$\log\left(\frac{(x - y)}{(y - z)}\right) = \log c_2$$

$$\frac{(x - y)}{(y - z)} = c_2$$

Result: $v = \dfrac{(x - y)}{(y - z)}$

The general solution is $\phi\left(xy + yz + zx, \dfrac{(x - y)}{(y - z)}\right) = 0$

5. Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$ **[AU N/D 2008 , M/J 2013]**

Solution:

Given $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

Lagrange's type $Pp + Qq = R$

The S.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$$

using x, y,-1 as lagrange's multipliers we get each of ratio

$$= \frac{x\,dx + y\,dy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

$$x\,dx + y\,dy - dz = 0$$

$$\int x\,dx + \int y\,dy - \int dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} - z = c_1$$

$$x^2 + y^2 - 2z = u$$

Taking the Lagrange's multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ *we get*

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(y^2 + z) - (x^2 + z) + (x^2 - y^2)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$$v = xyz$$

Result: Solution of given PDE is $\phi(x^2 + y^2 - 2z, xyz) = 0$

6. Solve $(x^2 - y^2 - z^2)p + 2xyq - 2xz = 0$ **[AU N/D 2001]**

Solution:

Given $(x^2 - y^2 - z^2)p + 2xyq - 2xz = 0$

Lagrange's type $Pp + Qq = R$

The S.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{(x^2 - y^2 - z^2)} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking 2nd and 3rd member we get

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log\left(\frac{y}{z}\right) = \log c_1$$

$$\frac{y}{z} = c_1$$

$$u = \frac{y}{z}$$

using x, y, z as lagrange's multiplier we get each of above ratio is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2} = \frac{dz}{2zx}$$

$$\frac{2(xdx+ydy+zdz)}{x[x^2-y^2-z^2+2y^2+2z^2]}=\frac{dz}{zx}$$

$$\frac{d(x^2+y^2+z^2)}{x(x^2+y^2+z^2)}=\frac{dz}{zx}$$

$$\int\frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}=\int\frac{dz}{z}$$

$$\log(x^2+y^2+z^2)=\log z+\log c_2$$

$$\log\left(\frac{x^2+y^2+z^2}{z}\right)=\log c_2$$

$$\frac{x^2+y^2+z^2}{z}=c_2, \qquad v=\frac{x^2+y^2+z^2}{z}$$

Result: The solution of given PDE is $\phi\left(\frac{y}{z},\frac{x^2+y^2+z^2}{z}\right)=0$

7. Solve ($(y-xz)p+(yz-x)q=(x+y)(x-y)$ **[AU M/J 2004, 2005]**

Solution:

Given $(y-xz)p+(yz-x)q=(x+y)(x-y)$

Lagrange's type $Pp+Qq=R$

The S.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{(y-xz)}=\frac{dy}{(yz-x)}=\frac{dz}{(x^2-y^2)}$$

Choosing x, y, z as lagrange's multipliers, we get each above ratio is equal to

$$=\frac{x\,dx+y\,dy+z\,dz}{xy-x^2z+y^2z-xy+zx^2-zy^2}$$

$$x\,dx+ydy+z\,dz=0$$

$$\int x\,dx+\int y\,dy+\int z\,dz=0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c$$

$$x^2+y^2+z^2=c_1 \quad ,c_2=2c$$

$$u=x^2+y^2+z^2$$

Choosing $y,x,1$ as Lagrange's Multiplier, we get

$$=\frac{ydx+xdy+dz}{y^2-xyz+xyz-x^2+x^2-y^2}$$

$$y\,dx+x\,dy+dz=0 \;;\; d(xy)+dz=0$$

$$d(xy+Z)=0$$

$$\int d(xy+z)=0$$

$$xy + z = c_2 \;;\; v = xy + Z$$

Result: The required solution is $\phi(x^2 + y^2 + z^2, xy + z) = 0$

8. Solve $pzx + qzy = y^2 - x^2$ **[AU N/D 2008, A/M 2009]**

Solution:

Given $pzx + qzy = y^2 - x^2$

Lagrange' s type $Pp + Qq = R$

$$\frac{dx}{zx} = \frac{dy}{zy} = \frac{dz}{y^2 - x^2}$$

Consider $\frac{dx}{zx} = \frac{dy}{zy}$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$`\log\left(\frac{x}{y}\right) = \log c_1$$

$$\left(\frac{x}{y}\right) = c_1$$

$$u = \left(\frac{x}{y}\right)$$

choose x,-y, z as a lagrange' s multiplier s, then the ratio are equal to

$$\frac{x\,dx - y\,dy + z\,dz}{zx^2 - zy^2 + zy^2 - zx^2} = \frac{xdx - ydy + zdz}{0}$$

$$0 = x\,dx - y\,dy + z\,dz$$

$$0 = \int x\,dx - \int y\,dy + \int z\,dz$$

$$c_2 = \frac{x^2}{2} - \frac{y^2}{2} + \frac{z^2}{2}$$

$$2c_2 = x^2 - y^2 + z^2$$

$$x^2 - y^2 + z^2 = c_2$$

$$v = x^2 - y^2 + z^2$$

Result: The Solution of given PDE is $\phi\left(\frac{x}{y}, x^2 - y^2 + z^2\right) = 0$

9. Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$ **[AU N/D 2016]**

Solution:

Given $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$

Lagrange's type $Pp+Qq=R$

Formula The S.E is $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$

$$\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)}$$

Choosing x, y, z as a Lagrange's multipliers, we get each of ratio are equal to

$$\frac{xdx+ydy+zdz}{x^2z^2-x^2y^2+y^2x^2-y^2z^2+z^2y^2-z^2x^2}=\frac{xdx+ydy+zdz}{0}$$

$$\int(xdx+ydy+zdz)=0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_1$$

$$u=x^2+y^2+z^2$$

Choosing $\dfrac{1}{x},\dfrac{1}{y},\dfrac{1}{z}$ as Lagrange's Multipliers, we get

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{z^2-y^2+x^2-z^2+y^2-x^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$$\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$$

$$\log x+\log y+\log z=\log c_2$$

$$\log(xyz)=\log c_2;xyz=c_2$$

$$v=xyz$$

Result: The solution of given PDE is $\varphi(x^2+y^2+z^2,xyz)=0$

10. Solve $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$ **[AU N/D 2010]**

Solution:

Given $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

Lagrange's type $Pp+Qq=R$

The S.E is $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

Choosing $\dfrac{1}{x},\dfrac{1}{y},\dfrac{1}{z}$ as Lagrange's Multipliers, we get

We get each ratio in (1),

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

Hence, $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

$$\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$$

$$\log x+\log y+\log z=\log a$$

$$\log(xyz)=\log a$$

$$\therefore a=xyz$$

Taking the Lagrange's multiples are $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$

We get each ratio in (1),

$$\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)}=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}$$

Hence, $\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz=0$

$$\int\frac{1}{x^2}dx+\int\frac{1}{y^2}dy+\int\frac{1}{z^2}dz=b$$

$$\int x^{-2}dx+\int y^{-2}dx+\int z^{-2}dx=b$$

$$\frac{x^{-1}}{-1}+\frac{y^{-1}}{-1}+\frac{z^{-1}}{-1}=b$$

$$x^{-1}+y^{-1}+z^{-1}=-b\Rightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=b$$

Result: Hence, the general solution is $f(a,b)=0$

i.e, $f\left(xyz,\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)=0$, where f is arbitrary

11. Solve the partial differential equation $(x-2z)p+(2z-y)q=y-x$

Solution: **[AU N/D 2011,17]**

Given $(x-2z)p+(2z-y)q=y-x$

This equation is of the form lagrange's linear equation

Lagrange' s type $Pp + Qq = R$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x-2z}=\frac{dy}{2z-y}=\frac{dz}{y-x} \qquad \text{….. (1)}$$

taking the lagrange's mulipliers are 1,1,1 *we get*

$$\text{each ratio in (1)} = \frac{dx+dy+dz}{\text{x - 2z} + \text{2z - y} + \text{y - x}} = \frac{dx+dy+dz}{0}$$

hence $d(x+y+z)=0$

integrating we get $x+y+z=a$

Taking the lagrange's mulipliers are $y, \text{x}, \text{2z}$ we get

$$each\,ratio\,in\,(1) = \frac{ydx+xdy+2zdz}{\text{xy - 2zy} + \text{2xz - xy} + \text{2zy - 2zx}} = \frac{ydx+xdy+2zdz}{0}$$

Hence, $ydx+xdy+2zdz=0$

i.e, $d(xy)+2zdz=0$ $[\because d(xy)=xdx+ydy]$

Integrating we get

$$\text{xy}+\frac{2\text{z}^2}{2}=b \Rightarrow \text{xy}+\text{z}^2=b$$

Result: Hence the general solution is $f(a,b)=0$

$$f(x+y+z, xy+\text{z}^2)=0$$

12. Find the general solution of $x(y^2-z^2)+y(z^2-x^2)q=z(x^2-y^2)$

Solution: **[AU A/M 2001]**

Given $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$ ….. (1)

Lagrange's type $\text{Pp}+\text{Qq}=\text{R}$

$$\frac{\text{dx}}{\text{P}}=\frac{\text{dy}}{\text{Q}}=\frac{\text{dz}}{R}$$

$$\frac{\text{dx}}{x(y^2-z^2)}=\frac{\text{dy}}{y(z^2-x^2)}=\frac{\text{dz}}{(x^2-y^2)} \quad \text{….. (2)}$$

Taking the Lagrange's multipliers are x, y, z we get each ratio in (2)

$$=\frac{xdx+ydy+zdz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}=\frac{xdx+ydy+zdz}{0}$$

$$xdx+ydy+zdz=0$$

Integrating we get, $\int xdx+\int ydy+\int zdz=0$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{a}{2}$$

$$x^2+y^2+z^2=a$$

Use the Lagrange's multipliers $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ we get

Each ratio in (2)

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$i.e, \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating we get, $\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$$i.e, xyz = b$$

Result: Hence the general solution is $f(a,b) = 0$

$i.e,\ f(x^2 + y^2 + z^2, xyz) = 0$ Where f is arbitrary.

13. Solve: $(y^2+z^2)p - xyq + xz = 0$ **[AU N/D 2013]**

Solution:

Given: $(y^2+z^2)p - xyq + xz$ ….. (1)

Lagrange's type $Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-xz} \quad \text{….. (2)}$$

Taking the Lagrange's multipliers are x, y, z we get each ratio in (2)

$$= \frac{xdx + ydy + zdz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{xdx + ydy + zdz}{0}$$

$$xdx + ydy + zdz = 0$$

Integrating we get, $\int xdx + \int ydy + \int zdz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$$

$$x^2 + y^2 + z^2 = a$$

$$x^2 + y^2 + z^2 = a$$

Taking 2nd and 3rd member , we get

Each ratio in (2)

$$\frac{dy}{-xy} = \frac{dz}{-xz}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating we get, $\int \frac{1}{y}dy = \int \frac{1}{z}dz$

$$\log y = \log z + \log b$$

$$\log\left(\frac{y}{z}\right) = \log b$$

$$i.e, \frac{y}{z} = b$$

Result: Hence the general solution is $f(a,b) = 0,$

$$i.e,\ f(x^2 + y^2 + z^2, \frac{y}{z}) = 0 \text{ Where f is arbitrary.}$$

14. Solve the Lagrange's equation $(x+2z)p + (2xz - y)q = x^2 + y.$

Solution: **[AU M/J 2014]**

Lagrange's type $\mathrm{Pp} + \mathrm{Qq} = \mathrm{R}$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x + 2z} = \frac{dy}{2xz - y} = \frac{dz}{x^2 + y}$$

Choose multipliers x,-1,-1

$$\frac{x dx - dy - dz}{x^2 + 2xz - 2xz + y - x^2 - y} = \frac{x dx - dy - dz}{0}$$

$$x dx - dy - dz = 0$$

$$\int x dx - \int dy - \int dz = 0$$

$$\frac{x^2}{2} - y - z = C_1$$

Choose Multipliers y,x,-2z

$$\frac{y dx + x dy - 2z dz}{yx + 2yz + 2x^2 z - xy - 2zx^2 - 2zy} = \frac{y dx + x dy - 2z dz}{0}$$

$$y dx + x dy - 2z dz = 0$$

$$\int y dx + \int x dy - \int 2z dz = 0$$

$$xy - xy - z^2 = C_2$$

$$z^2 = C_2$$

Result:

The general solution is

$$\therefore \quad \varphi(C_1, C_2) = 0$$

$$\varphi\left(\frac{x^2}{2} - y - z, z^2\right) = 0$$

15. Find the general solution of $(z^2 - y^2 - 2yz)p + (xy + zx)q = (xy - zx)$

Solution: **[AU N/D 2015, A/M 2017]**

Given $(z^2 - y^2 - 2yz)p + (xy + zx)q = (xy - zx)$ ….. (1)

Lagrange's type $\mathrm{Pp} + \mathrm{Qq} = \mathrm{R}$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$(z^2 - y^2 - 2yz)p + (xy + zx)q = (xy - zx)$$

$$\frac{dx}{(z^2 - y^2 - 2yz)} = \frac{dy}{(xy + zx)} = \frac{dz}{(xy - zx)} \qquad \text{..... (2)}$$

Taking the Lagrange's multipliers are x, y, z we get each ratio in (2)

$$= \frac{xdx + ydy + zdz}{xz^2 - xy^2 - 2xyz + xy^2 + xyz + xyz - xz^2} = \frac{xdx + ydy + zdz}{0}$$

$$xdx + ydy + zdz = 0$$

Integrating we get, $\int xdx + \int ydy + \int zdz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2 + y^2 + z^2 = u$$

Taking 2^{nd} and 3^{rd} ratio

$$\frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

$$\frac{dy}{y + z} = \frac{dz}{y - z}$$

$$ydy - zdy = ydz + zdz$$

$$ydy - zdz = d(yz)$$

Integrating,

$$\int ydy - \int zdz = \int d(yz)$$

$$i.e, \frac{y^2}{2} - \frac{z^2}{2} = yz + c_2$$

$$2yz - y^2 + z^2 = v$$

Result: Hence the general solution is $f(u, v) = 0$

$$i.e,\ f(x^2 + y^2 + z^2, 2yz - y^2 + z^2) = 0$$

Homogeneous Partial Differential Equation

16. Solve $(D^2 - DD' - 30D'^2)y = xy + e^{6x+y}$ **[AU M/J 2004, 2009]**

Solution:

$$\text{Auxillary Equation is } m^2 - m - 30 = 0$$

$$(m - 6)(m + 5) = 0$$

$$m = 6, -5$$

$$\text{Complementary function} = f_1(y + 6x) + f_2(y - 5x)$$

$$\text{Particular Integral} = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{(D^2 - DD' - 30D'^2)} xy$$

$$= \frac{1}{D^2\left(1-\left(\frac{D'}{D}+\frac{30D'^2}{D^2}\right)\right)}xy$$

$$= \frac{1}{D^2}\left(1-\left(\frac{D'}{D}+\frac{30D'^2}{D^2}\right)\right)^{-1}xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{D'}{D}+\frac{30D'^2}{D^2}\right)+\left(\frac{D'}{D}+\frac{30D'^2}{D^2}\right)^2+\ldots\ldots\right]xy$$

$$= \frac{1}{D^2}\left(1+\frac{D'}{D}\right)xy$$

$$= \frac{1}{D^2}\left(xy+\frac{x}{D}\right)$$

$$= \frac{1}{D^2}(xy)+\frac{1}{D^3}x$$

$$= y\frac{x^3}{6}+\frac{x^4}{24}$$

$$P.I_2 = \frac{1}{(D^2-DD'-30D'^2)}e^{(6x+y)}$$

$$= \frac{1}{36-6-30}e^{(6x+y)}$$

Condition fails

$$= \frac{x}{(2D-D')}e^{(6x+y)}$$

$$= \frac{x}{12-1}e^{(6x+y)}$$

$$= \frac{x}{11}e^{(6x+y)}$$

$$P.I = y\frac{x^3}{6}+\frac{x^4}{24}+\frac{x}{11}e^{(6x+y)}$$

Result: The Complete solution is

$$z = c.f + P.I$$

$$= f_1(y+6x)+f_2(y-5x)+y\frac{x^3}{6}+\frac{x^4}{24}+\frac{x}{11}e^{(6x+y)}$$

17. Solve $(D^2-6DD'-5D'^2)z = e^x \sinh y + xy$ **[AU N/D 2006 , A/M 1997]**

Solution:

Auxillary Equation is $m^2-6m+5=0$

$$(m-1)(m-5)=0$$
$$m=1,5$$

Complementary function is

$$C.F = f_1(y+6x)+f_2(y+5x)$$

Particular Integral = $P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D^2-6DD'-5D'^2} e^x \sinh y$$
$$= \frac{1}{D^2-6DD'-5D'^2} e^x \left(\frac{e^y-e^{-y}}{2}\right)$$
$$= \frac{1}{2}\frac{1}{D^2-6DD'-5D'^2}\left(e^{(x+y)}-e^{(x-y)}\right)$$
$$= \frac{1}{2}\frac{1}{D^2-6DD'-5D'^2}e^{(x+y)} - \frac{1}{2}\frac{1}{D^2-6DD'-5D'^2}e^{(x-y)}$$
$$= \frac{1}{2}\frac{1}{(1-6-5)}e^{(x+y)} - \frac{1}{2}\frac{1}{(1+6-5)}e^{(x-y)}$$
$$= \frac{1}{-20}e^{(x+y)} - \frac{1}{4}e^{(x-y)}$$
$$P.I_1 = \frac{1}{-20}e^{(x+y)} - \frac{1}{4}e^{(x-y)}$$
$$P.I_2 = \frac{1}{D^2-6DD'-5D'^2}\, xy$$
$$= \frac{1}{D^2\left(1-\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)}\, xy$$
$$= \frac{1}{D^2}\left(1-\left(\frac{6D'}{D}+\frac{5D'^2}{D^2}\right)\right)^{-1} xy$$
$$= \frac{1}{D^2}\left(1+\frac{6D'}{D}+\frac{5D'^2}{D^2}\right)xy$$
$$= \frac{1}{D^2}\left(xy+\frac{6x}{D}\right)$$
$$= \frac{yx^3}{6}+\frac{6x^4}{24}$$

Particular Integral = $P.I_1 + P.I_2$

$$= -\frac{1}{20}e^{(x+y)} - \frac{1}{4}e^{(x-y)} + \frac{yx^3}{6} + \frac{6x^4}{24}$$

Result: The Complete Solution is

$$Z = C.F + P.I$$

$$= f_1(y+6x)+f_2(y-5x)- \frac{1}{20}e^{(x+y)} - \frac{1}{4}e^{(x-y)} + \frac{yx^3}{6} + \frac{6x^4}{24}$$

18. Solve $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x+6y)$ **[AU N/D 2010, A/M 2009]**

Solution:

Given $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x+6y)$

Auxillary equation is $m^3 - 4m^2 + 4m = 0$

$$m(m^2 - 4m + 4) = 0$$

$$m = 0,2,2$$

$$C.F = f_1(y) + f_2(y+2x) + xf_3(y+2x)$$

Particular Integral

$$P.I = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 6\sin(3x+6y)$$

$$= 6\frac{1}{-9D - 4(-9)D' - 144D} \sin(3x+6y)$$

$$= 6\frac{1}{-153D + 36D'} \sin(3x+6y)$$

$$= \frac{6}{9}\frac{1}{4D' - 17D} \sin(3x+6y)$$

$$= \frac{2}{3}\frac{(4D' + 17D)}{(16D'^2 - 289D^2)} \sin(3x+6y)$$

(multiply and divide by $(4D' + 17D)$)

$$= \frac{2}{3}\frac{(4D' + 17D)}{(-576 + 2601)} \sin(3x+6y)$$

$$= \frac{2}{6075}(4D' + 17D)\sin(3x+6y)$$

$$= \frac{2}{6075}\left[24\cos(3x+6y) + 51\cos(3x+6y)\right]$$

$$= \frac{2\times 75}{6075}\cos(3x+6y)$$

$$= \frac{6}{243}\cos(3x+6y)$$

$$P.I = \frac{2}{81}\cos(3x+6y)$$

Result:

Hence the general solution is Z = C.F+P.I

$$Z = f_1(y) + f_2(y+2x) + xf_3(y+2x) + \frac{2}{81}\cos(3x+6y)$$

19. Solve $r+s-6t = y\cos x \text{ (or)} (D^2 + DD' - 6D'^2)Z = y\cos x$

Solution: **[AU M/J 2013, 2014, N/D 2016, A/M 2018]**

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial xy} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

$$(D^2 + DD' - 6D'^2)Z = y\cos x$$

Auxillary equation is

$$m^2 + m - 6 = 0$$
$$(m+3)(m-2) = 0$$
$$m = -3, 2$$

Complement ary function,

$$f_1(y-3x) + f_2(y+2x)$$

Particular Integral

$$P.I = \frac{1}{D^2 + DD' - 6D'^2}\, y\cos x$$
$$= \frac{1}{(D-2D')(D+3D')}\, y\cos x$$
$$= \frac{1}{D-2D'}\int (C+3x)\cos x\, dx$$
$$= \frac{1}{D-2D'}\left[(c+3x)\sin x - 3(-\cos x)\right]$$
$$= \frac{1}{D-2D'}\left[(c+3x)\sin x + 3\cos x\right]$$
$$= \frac{1}{D-2D'}[\,y\sin x + 3\cos x]$$
$$= \int (c_1 - 2x)\sin x + 3\cos x\, dx$$
$$= \int c_1 \sin x\, dx - 2\int x\sin x\, dx + 3\int \cos x\, dx$$
$$= -c_1\cos x + 3\sin x - 2\int x\sin x\, dx$$
$$= -c_1\cos x + 3\sin x - 2\int x\, d(-\cos x)$$
$$= -c_1\cos x + 3\sin x - 2\left[-x\cos x + \int \cos x\, dx\right]$$
$$= -c_1\cos x + 3\sin x + 2x\cos x - 2\sin x$$
$$= -y\cos x - 2x\cos x + 3\sin x + 2x\cos x - 2\sin x$$
$$P.I = \sin x - y\cos x$$

Result: Complete solution is

$$z = C.F + P.I$$
$$z = f_1(y-3x) + f_2(y+2x) + \sin x - y\cos x$$

20.Solve $\left(D^2 - 5DD^1 + 6D^{12}\right)z = y\sin x$ **[AU N/D 2017]**

Solution:

Given PDE is

$$\left(D^2-5DD'+6D'^2\right)Z=y\sin x$$

Complete Solution= CF+PI

To find CF:

The Auxiliary equation is

$$m^2+5m+6=0$$
$$(m+2)(m+3)=0$$
$$m=-2,-3$$

Therefore

$$C.F=f_1(y-2x)+f_2(y-3x)$$

To find PI

$$P.I=\frac{1}{\left(D^2+5DD'+6D'^2\right)}y\sin x$$

$$=IPe^{ix+oy}\frac{1}{D^2-5DD'+6D'^2}y$$

$$=IPe^{ix+oy}\frac{1}{(D+i)^2-5(D+i)D'+6D'^2}y$$

$$=IPe^{ix+0y}\frac{1}{-1+D^2+2iD-5DD'-5iD'+6D'^2}y$$

$$=(-1)IP(\cos x+i\sin x)\left[1-\left(D^2+2iD-5DD'-5iD'+6D'^2\right)\right]^{-1}y$$

:

$$=(-1)IP(\cos x+i\sin x)\left[1+D^2+2iD-5iD'-5DD'+6D'^2\right]y$$

$$=(-1)IP(\cos x+i\sin x)[y-5i]$$

$$=5\cos x-y\sin x$$

Result: The general solution is

$$z=C.F+P.I$$
$$=f_1(y-2x)+f_2(y-3x)+5\cos x-y\sin x$$

21. Solve $(D^3-2D^2D')z=2e^{2x}+3x^2y$ **[AU N/D 2001]**

Solution:

The auxiliary equation is

$$(m^3-2m^2)=0$$
$$m^2(m-2)=0$$
$$m=0,0,2$$

$$C.F=\phi_1(y)+x\phi_2(y)+\phi_3(y+2x)$$

$$P.I_1 = \frac{1}{(D^3 - 2D^2D^{'})} 2e^{2x+0y} \ (replace \ \ D \ by \ 2 \text{ and } D^{'} \text{ by } 1)$$

$$= 2\frac{1}{8}e^{2x} = \frac{1}{4}e^{2x}$$

$$P.I_2 = \frac{1}{(D^3 - 2D^2D^{'})} 3x^2y$$

$$= 3\frac{1}{D^2(D-2D^{'})}x^2y$$

$$= 3\frac{1}{D^3}\frac{1}{\left(1-\frac{2D^{'}}{D}\right)}x^2y$$

$$= 3\frac{1}{D^3}\left(1-\frac{2D^{'}}{D}\right)^{-1}x^2y$$

$$= 3\frac{1}{D^3}\left[1+\frac{2D^{'}}{D}+\left(\frac{2D^{'}}{D}\right)^2+\dots\right]x^2y$$

$$= 3\frac{1}{D^3}\left[x^2y+2\frac{1}{D}x^2\right] = 3\frac{1}{D^3}\left[x^2y+2\frac{x^3}{3}\right]$$

$$= 3\frac{1}{D^2}\left(\frac{x^3y}{3}+\frac{2}{3}\frac{x^4}{4}\right) = 3\frac{1}{D}\left(\frac{x^4y}{12}+\frac{1}{6}\frac{x^5}{5}\right)$$

$$= 3\left(\frac{x^5y}{60}+\frac{1}{30}\frac{x^6}{6}\right) = \frac{x^5y}{20} + \frac{x^6}{60}$$

Result: $\therefore z = C.F + PI_1 + PI_2$

$$= \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) + \frac{1}{4}e^{2x} + \frac{x^5y}{20} + \frac{x^6}{60}$$

22. Solve the equation $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right)z = \cos(2x+y)$ **[AU N/D 2010]**

Solution:

Given $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right)z = \cos(2x+y)$

The auxiliary equation is

$$m^3 + m^2 - 4m - 4 = 0 \text{ replace D by m D' by1}$$

$$m^2(m+1) - 4(m+1) = 0$$

$$(m+1)(m^2-4) = 0$$

$$(m+1)(m+2)(m-2) = 0$$

$$m = -1, m = -2, m = 2$$

$$C.F = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x)$$

$$P.I = \frac{1}{D^3 + D^2D' - 4DD'^2 - 4D'^3}\cos(2x+y) \text{ (replace } D^2 \text{by } -2^2, DD' \text{ by } -2, D'^2 \text{ by } -1)$$

$$= \frac{1}{-4D - 4D' + 4D + 4D'}\cos(2x+y)$$

$$= \frac{1}{0}\cos(2x+y) \qquad \text{(ordinary rule fails)}$$

$$= x\frac{1}{3D^2 + 2DD' - 4D'^2}\cos(2x+y) \text{ (replace } D^2 \text{ by } -2^2, DD' \text{ by } -2, D'^2 \text{ by } -1)$$

$$= x\frac{1}{-12-4+4}\cos(2x+y) = \frac{-x}{12}\cos(2x+y)$$

Result: The general solution is z=C.F+P.I

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+2x) - \frac{x}{12}\cos(2x+y)$$

23. Solve the equation $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x+2y) + x$

Solution: **[AU N/D 2011,2012]**

Given $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x+2y) + x$

The Auxiliary equation is

$$m^3 - 7m - 6 = 0$$

Put $m = 1$, $we\ get 1 - 7 - 6 \neq 0$

Put $m = -1$, $we\ get -1 + 7 - 6 = 0$

$$\begin{array}{c|cccc} -1 & 0 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & & & & 0 \end{array}$$

$\therefore \quad m = -1$ is a root

Remaining equation is

$$m^2 - m - 6 = 0$$

$$i.e, m+1 = 0, m^2 - m - 6 = 0$$

$$\therefore \quad m = -1, -2, 3$$

$$C.F = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$$

$$P.I_1 = \frac{1}{[D^3 - 7DD'^2 - 6D'^3]}\cos(x+2y)$$

$replace\, D^2 by -1^2 = -1$

$DD' by -(1)(2) = -2$

$DD' by -(1)(2) = -2$

$$= \frac{1}{-D - 7(-2)D' - 6(-4)D'}\cos(x+2y)$$

$$= \frac{1}{-D + 14D' + 24D'}\cos(x+2y)$$

$$= \frac{38D' + D}{(38D' - D)(38D' + D)}\cos(x+2y)$$

$$= \frac{1}{1444D'^2 - D^2}\left[-76\sin(x+2y) - \sin(x+2y)\right]$$

$$=\frac{1}{1444D'^2-D^2}\left[-77\sin(x+2y)\right]$$

$$=-77\frac{1}{1444(-4)-(-1)}\sin(x+2y)=\frac{77}{5775}\sin(x+2y)$$

$$P.I_1=\frac{1}{75}\sin(x+2y)$$

$$P.I_2=\frac{1}{[D^3-7DD'^2-6D'^3]}x=\frac{1}{D^3\left[1-\frac{7D'^2}{D^2}-\frac{6D'^3}{D^3}\right]}x$$

$$=\frac{1}{D^3}\left[1-\frac{7D'}{D^2}-\frac{6D'^3}{D^3}\right]^{-1}x=\frac{1}{D^3}\left[1+\left(\frac{7D'}{D^2}-\frac{6D'^3}{D^3}\right)+\cdots\right]x$$

$$=\frac{1}{D^3}\left[x+\frac{7}{D^2}(0)+\cdots\right]=\frac{1}{D^3}x=\frac{x^4}{24}$$

Result:

Hence, the general solution is z = C.F + P.I

$$z=\phi_1(\mathrm{y}-x)+\phi_2(\mathrm{y}-2x)+\phi_3(\mathrm{y}+3x)+\frac{x^4}{24}+\frac{1}{75}\sin(x+2y)$$

24. Solve $\left(D^2-DD'-2D'^2\right)z=2x+3y+e^{2x+4y}$ **[AU N/D 2013]**

Solution:

The auxiliary equation is $m^2-m-2=0$

$$(m+1)(m-2)=0$$

$$m=-1,\ 2$$

$$C.F=f_1(y-x)+f_2(y+2x)$$

To find P.I

$$P.I_1=\frac{1}{D^2-DD'-2D'^2}(2x+3y)$$

$$=\frac{1}{D^2\left[1-\left(\frac{D'}{D}+\frac{2D'^2}{D^2}\right)\right]}(2x+3y)$$

$$=\frac{1}{D^2}\left[1-\left(\frac{D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{D'}{D} + \frac{2D'^2}{D^2}\right)^2 + \ldots\right](2x+3y)$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D}\right](2x+3y)$$

$$= \frac{1}{D^2}\left[2x+3y + \frac{D'}{D}(2x+3y)\right]$$

$$= \frac{1}{D^2}\left[2x+3y+\frac{3}{D}\right]$$

$$= \frac{1}{D}\left[\frac{2x^2}{2} + 3xy + \frac{3x^2}{2}\right]$$

$$P.I_1 = \frac{5x^3 + 9x^2 y}{6}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 2D'^2}\, e^{2x+4y} \qquad \begin{bmatrix} D \to 2 \\ D' \to 4 \end{bmatrix}$$

$$= \frac{e^{2x+4y}}{4 - 8 - 32}$$

$$P.I_2 = -\frac{e^{2x+4y}}{36}$$

Result: $\therefore$ Z = C.F + P.I

$$Z = f_1(y - x) + f_2(y + 2x) + \frac{5x^3 + 9x^2 y}{6} - \frac{e^{2x+4y}}{36}$$

25. Solve: $\left(D^2 - 2DD'\right)z = x^3 y + e^{2x-y}$ **[AU N/D 2014]**

Solution:

The A.E is $m^2 - 2m = 0$

$$m(m-2) = 0$$

$$m = 0,2$$

$$\therefore \quad C.F = f_1(y + 0x) + f_2(y + 2x)$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - 2DD'}\, x^3 y$$

$$= \frac{1}{D^2\left[\frac{D^2 - 2DD'}{D^2}\right]}\, x^3 y$$

$$= \frac{1}{D^2\left[1 - \frac{2D'}{D}\right]}\, x^3 y$$

$$=\frac{1}{D^2}\left[1-\frac{2D'}{D}\right]^{-1}x^3y$$

$$=\frac{1}{D^2}\left[1+\frac{2D'}{D}+\frac{4D'^2}{D^2}+.........\right]x^3y$$

$$=\frac{1}{D^2}\left[x^3y+\frac{2}{D}(x)^2\right]$$

$$=\frac{1}{D^2}\left(x^3y\right)+\frac{2}{D^3}(x)^3$$

$$=\frac{x^5y}{20}+\frac{2x^6}{120}$$

$$=\frac{x^5y}{20}+\frac{x^6}{60}$$

$$P.I_2=\frac{1}{D^2-2DD'}e^{2x-y}$$

$$=\frac{1}{(2)^2-2(2)(-1)}e^{2x-y}$$

$$=\frac{1}{4+4}e^{2x-y}$$

$$=\frac{1}{8}e^{2x-y}$$

Result: $\therefore \quad z=C.F+P.I_1+P.I_2$

$$=f_1(y+0x)+f_2(y+2x)+\frac{yx^5}{20}+\frac{x^6}{60}+\frac{e^{2x-y}}{8}$$

26. Solve the equation $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)$

Solution: **[AU N/D 2011,2012,2014]**

Given $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)$

Put $m=1,\ we\ get 1-7-6\neq 0$

Put $m=-1,\ we\ get -1+7-6=0$

$\therefore m=-1$ is a root

$$\begin{array}{r|rrrr} -1 & 0 & 0 & -7 & -6 \\ & 0 & -1 & 1 & 6 \\ \hline & & & & 0 \end{array}$$

$$i.e, m+1=0$$
$$m=-1$$

Remaining equation is $m^2-m-6=0$

$$(m+2)(m-3)=0$$
$$m=-2,\ 3$$
$$\therefore\quad m=-1,-2,3$$
$$\therefore\quad C.F=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$$

$$P.I_1=\frac{1}{[D^3-7DD'^2-6D'^3]}\ \sin(x+2y)$$
$$=\frac{1}{-D+28D+24D'}\ \sin(x+2y)$$
$$=\frac{1}{27D+24D'}\sin(x+2y)$$
$$=\frac{D}{27D^2+24DD'}\sin(x+2y)$$
$$=\frac{D\sin(x+2y)}{-27-48}$$
$$=\frac{\cos(x+2y)}{-75}$$

replace D^2 by $-1^2=-1$

DD' by $-(1)(2)=-2$

DD' by $-(1)(2)=-2$

Result:

z = C.F + P.I

$$z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)-\frac{1}{75}\cos(x+2y)$$

27.Solve the equation $(D^2+4DD'-5D'^2)z=\sin(x-2y)+e^{2x-y}$

Solution: **[AU A/M 2018]**

Given $(D^2+4DD'-5D'^2)z=\sin(x-2y)+e^{2x-y}$

$A.E$ is $m^2+4m-5=0$

$$(m+5)(m-1)=0$$
$$m=-5,1$$
$$\therefore\quad m=-5,1$$
$$\therefore\quad C.F=\varphi_1(y-5x)\ +\varphi_2(y+x)$$

$$P.I_1=\frac{1}{[D^2+4DD'-5D'^2]}\ \sin(x-2y)$$
$$=\frac{1}{-1+4(2)-5(-4)}\ \sin(x-2y)$$
$$=\frac{\sin(x-2y)}{27}$$

replace D^2 by $-1^2=-1$

DD' by $-(1)(-2)=2$

replace D'^2 by $-2^2=-4$

$$\text{P.I}_2 = \frac{e^{2x-y}}{[D^2+4DD'-5D'^2]}$$

$$= \frac{e^{2x-y}}{[2^2+4(2)(-1)-5(-1)^2]}$$

$$= \frac{e^{2x-y}}{-9}$$

Result:

z = C.F + P.I

$$z = \varphi_1(\text{y-5}x) + \varphi_2(\text{y+}x) + \frac{1}{27}\sin(x-2y) - \frac{e^{2x-y}}{9}$$

28. Solve: $(D^2-3DD'-2D'^2)z = (2+4y)e^{x+2y}$ **[AU A/M 2015]**

Solution:

Given $(D^2-3DD'-2D'^2)z = (2+4y)e^{x+2y}$

The A.E is $m^2-3m+2=0$

$$(m-1)(m-2)=0$$

$$m=1,2$$

$$\therefore \quad C.F = f_1(y+x)+f_2(y+2x)$$

$$\text{P.I} = \frac{1}{[D^2-3DD'+2D'^2]}\ (2+4x)e^{x+2y}$$

$$= e^{x+2y}\frac{1}{(D+1)^2-3(D+1)(D'+2)+2(D'+2)^2}\ (2+4x)$$

$$= e^{x+2y}\frac{1}{D^2+2D+1-3DD'-6D-3D'-6+2D'^2+8D'+8}\ (2+4x)$$

$$= e^{x+2y}\frac{1}{D^2-4D-3DD'+5D'+2D'^2+3}\ (2+4x)$$

$$= e^{x+2y}\frac{1}{D^2\left[1-\frac{4}{D}-\frac{3D'}{D}+\frac{5D'}{D^2}+\frac{2D'^2}{D^2}+\frac{3}{D^2}\right]}\ (2+4x)$$

$$= \frac{e^{x+2y}}{D^2}\left[1-\left(\frac{4}{D}+\frac{3D'}{D}-\frac{5D'}{D^2}-\frac{2D'^2}{D^2}-\frac{3}{D^2}\right)\right]^{-1}(2+4x)$$

$$= \frac{e^{x+2y}}{D^2}\left[1+\left(\frac{4}{D}+\frac{3D'}{D}-\frac{5D'}{D^2}-\frac{2D'^2}{D^2}-\frac{3}{D^2}\right)\right](2+4x)$$

$$= \frac{e^{x+2y}}{D^2}\left[(2+4x)+\frac{4}{D}(2+4x)-\frac{3}{D^2}(2+4x)\right]$$

$$= \frac{e^{x+2y}}{D^2}\left[2+4x+8x+32x^2-3x^2-2x^3\right]$$

$$= \frac{e^{x+2y}}{D^2}\left[2+12x+29x^2-2x^3\right]$$

$$= e^{x+2y}\left[x^2+2x^3+\frac{29}{12}x^4-\frac{x^5}{10}\right]$$

Result: *The general solution is* $z = C.F + P.I$

$$z = f_1(y+x)+f_2(y+2x)+e^{x+2y}\left[x^2+2x^3+\frac{29}{12}x^4-\frac{x^5}{10}\right]$$

29. Solve: $\left(D^3-2D^2D'\right)z = 2e^{2x}+3x^2y$ **[AU M/J 2016]**

Solution:

Given:

$$\left(D^3-2D^2D'\right)z = 2e^{2x}+3x^2y$$

The Auxillary equation is

$$m^3-2m^2=0$$

$$(m-2)m^2=0$$

$$m=0,0,2$$

$$\therefore\quad C.F = f_1(y+0x)+xf_2(y+0x)+f_3(y+2x)$$

$$\text{P.I}_1 = \frac{1}{[D^3-2D^2D']}\,2e^{2x+0y}$$

$$= \frac{1}{[8-0]}\,2e^{2x+0y}$$

$$= \frac{1}{4}\,e^{2x+0y}$$

$$P.I_2 = \frac{1}{D^3-2D^2D'}\,(3x^2y)$$

$$= \frac{3}{D^3\left[1-\frac{2D'}{D}\right]}\,x^2y$$

$$= \frac{3}{D^3}\left[1-\frac{2D'}{D}\right]^{-1}x^2y$$

$$= \frac{3}{D^3}\left[1+\frac{2D'}{D}+\frac{4D'^2}{D^2}\right]x^2y$$

$$= \frac{3}{D^3}\left[x^2 y + \frac{2x^2}{D}\right]$$

$$= \frac{3}{D^3}\left[x^2 y\right] + 6\frac{x^2}{D^4}$$

$$= 3y\frac{x^5}{60} + 6\frac{x^6}{360}$$

$$= \frac{x^5 y}{20} + \frac{x^6}{60}$$

Result: $\therefore\ Z = C.F + P.I_1 + P.I_2$

$$\therefore\quad Z = f_1(y+0x) + xf_2(y+0x) + f_3(y+2x) + \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

30. Solve $\left(D^2 + D^{1^2}\right)z = x^2 y^2$ **[AU N/D2015]**

Solution:

Given that

$$\left(D^2 + D^{1^2}\right)z = x^2 y^2$$

Auxiliary equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$C.F = f_1(y+ix) + f_2(y-ix)$$

$$\text{P.I} = \frac{1}{\text{D}^2 + \text{D}'^2} x^2 y^2$$

$$= \frac{1}{\text{D}^2\left[1 + \frac{\text{D}'^2}{\text{D}^2}\right]} x^2 y^2$$

$$= \frac{1}{\text{D}^2}\left[1 + \frac{\text{D}'^2}{\text{D}^2}\right]^{-1} x^2 y^2$$

$$= \frac{1}{\text{D}^2}\left[1 - \frac{\text{D}'^2}{\text{D}^2} + \frac{\text{D}'^4}{\text{D}^4} - \ldots\ldots\ldots\ldots\right] x^2 y^2$$

$$= \frac{1}{\text{D}^2}\left[x^2 y^2 - \frac{\text{D}'^2}{\text{D}^2}\left(x^2 y^2\right)\right]$$

$$= \frac{1}{\text{D}^2}\left[x^2 y^2 - \frac{2x^2}{\text{D}^2}\right]$$

$$= \frac{1}{D^2}\left[x^2y^2 - \frac{2x^4}{12}\right]$$
$$= \frac{1}{D^2}\left[x^2y^2 - \frac{x^4}{6}\right]$$
$$= \frac{1}{D}\left[\frac{x^3y^2}{3} - \frac{x^5}{30}\right]$$
$$= \left[\frac{x^3y^2}{12} - \frac{x^5}{180}\right]$$

Result: Complete solution

Z = Comp. function + Particular Integral

$$= f_1(y+ix) + f_2(y-ix) + \left[\frac{x^3y^2}{12} - \frac{x^5}{180}\right]$$

31. Solve $\left(D^2 + 2DD^1 + D^{1^2}\right)z = 2\cos y - x\sin y$ **[AU N/D 2015]**

Solution:

Given PDE is

$$\left(D^2 + 2DD' + D'^2\right)Z = 2\cos y - x\sin y$$

Complete Solution= CF+PI_1-PI_2

To find CF:

The Auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)(m+1) = 0$$
$$m = -1, -1$$

Therefore

$$C.F = f_1(y-x) + xf_2(y-x)$$

To find PI_1: $$PI_1 = \frac{1}{\left(D^2 + 2DD' + D'^2\right)}2\cos y$$
$$= \frac{1}{\left(0^2 + 2(0) - 1^2\right)}2\cos y$$
$$= -2\cos y$$

To find PI_2:

$$P.I_2 = \frac{1}{\left(D^2 + 2DD' + D'^2\right)}x\sin y$$
$$= \frac{1}{(D+D')^2}xIPe^{0x+iy}$$

$$= IPe^{0x+iy}\frac{1}{(D+D'+i)^2}x$$

$$= IPe^{0x+iy}\frac{1}{-1+D^2+D'^2+2iD+2iD'+2DD'}x$$

$$=(-1)IP(\cos y+i\sin y)\left[1-\left(D^2+D'^2+2iD+2iD'+2DD'\right)\right]^{-1}x$$

$$=(-1)IP(\cos y+i\sin y)[1+D^2+D'^2+2iD+2iD'+2DD']x$$

$$=(-1)IP(\cos y+i\sin y)[x+2i]$$

$$=-2\cos y-x\sin y$$

Result: The general solution is

$$z = C.F + P.I$$

$$= f_1(y-x)+xf_2(y-x)-2\cos y-2\cos y-x\sin y$$

$$= f_1(y-x)+xf_2(y-x)-4\cos y-x\sin y$$

32. Solve $(D^2+2DD'+D'^2)z = x^2y + e^{x-y}$ **[AU A/M 2017]**

Solution:

Given that

$$(D^2+2DD'+D'^2)z = x^2y + e^{x-y}$$

Auxiliary equation is $(m+1)^2 = 0$

$$m = -1,-1$$

$$C.F = f_1(y-x)+xf_2(y-x)$$

$$\text{P.I}_1 = \frac{1}{\text{D}^2+2D\text{D}'+\text{D}'^2}x^2y$$

$$\text{P.I}_1 = \frac{1}{\text{D}^2\left(\dfrac{\text{D}^2+2D\text{D}'+\text{D}'^2}{\text{D}^2}\right)}x^2y$$

$$= \frac{1}{\text{D}^2}\left[1+\left(\frac{2\text{D}'}{\text{D}}+\frac{\text{D}'^2}{\text{D}^2}\right)\right]^{-1}x^2y$$

$$= \frac{1}{\text{D}^2}\left[1-\left(\frac{2\text{D}'}{\text{D}}+\frac{\text{D}'^2}{\text{D}^2}\right)+\ldots\ldots\ldots\ldots\right]x^2y$$

$$= \frac{1}{\text{D}^2}\left[x^2y-\frac{2\text{D}'}{\text{D}}\left(x^2y\right)\right]$$

$$= \frac{1}{D^2}\left[x^2 y \;-\frac{2x^2}{D}\right]$$

$$= \frac{1}{D^2}\left[x^2 y \;-\frac{2x^3}{3}\right]$$

$$= \frac{1}{D^2}\left[x^2 y \;-\frac{2x^3}{3}\right]$$

$$= \frac{1}{D}\left[\frac{x^3 y}{3}-\frac{2x^4}{3*4}\right]$$

$$P.I_1 = \left[\frac{x^4 y}{12}-\frac{x^5}{30}\right]$$

$$P.I_2 = \frac{e^{x-y}}{D^2+2DD'+D'^2}$$

Put D= 1 and D' = -1

$$P.I_2 = \frac{e^{x-y}}{(1)^2+2(1)(-1)+(-1)^2} = \frac{e^{x-y}}{0}$$

$$P.I_2 = \frac{xe^{x-y}}{D+2D'} = \frac{xe^{x-y}}{2(1)+2(-1)} = \frac{xe^{x-y}}{0}$$

$$P.I_2 = \frac{x^2e^{x-y}}{2}$$

Result :

Z = Comp. function + Particular Integral

$$= f_1(y-x)+xf_2(y-x)+\left[\frac{x^4 y}{12}-\frac{x^5}{30}\right]+\frac{x^2e^{x-y}}{2}$$

33. Solve $\left(D^2+2DD'+D'^2\right)z = xy+e^{x-y}$ **[AU N/D 2017]**

Solution:

Given that

$$\left(D^2+2DD'+D'^2\right)z = xy+e^{x-y}$$

Auxiliary equation is $(m+1)^2 = 0$

$$m = -1,-1$$

$$C.F = f_1(y-x)+xf_2(y-x)$$

$$P.I_1 = \frac{1}{D^2+2DD'+D'^2}\,xy$$

$$\text{P.I}_1 = \frac{1}{D^2\left(\frac{D^2+2DD'+D'^2}{D^2}\right)} xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{2D'}{D}+\frac{D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{1}{D^2}\left[1-\left(\frac{2D'}{D}+\frac{D'^2}{D^2}\right)+.............\right]xy$$

$$= \frac{1}{D^2}\left[xy-\frac{2D'}{D}(xy)\right]$$

$$= \frac{1}{D^2}\left[xy-\frac{2x}{D}\right]$$

$$= \frac{1}{D^2}\left[xy-\frac{2x^2}{2}\right]$$

$$= \frac{1}{D^2}\left[xy-x^2\right]$$

$$= \frac{1}{D}\left[\frac{x^2y}{2}-\frac{x^3}{3}\right]$$

$$\text{P.I}_1 = \left[\frac{x^3y}{6}-\frac{x^4}{12}\right]$$

$$\text{P.I}_2 = \frac{e^{x-y}}{D^2+2DD'+D'^2}$$

Put D= 1 and D’ = -1

$$\text{P.I}_2 = \frac{e^{x-y}}{(1)^2+2(1)(-1)+(-1)^2} = \frac{e^{x-y}}{0}$$

$$\text{P.I}_2 = \frac{xe^{x-y}}{D+2D'} = \frac{xe^{x-y}}{2(1)+2(-1)} = \frac{xe^{x-y}}{0}$$

$$\text{P.I}_2 = \frac{x^2e^{x-y}}{2}$$

Result :

$$Z = \text{Comp. function} + \text{Particular Integral}$$

$$z = f_1(y-x)+xf_2(y-x)+\left[\frac{x^3y}{6}-\frac{x^4}{12}\right]+\frac{x^2e^{x-y}}{2}$$

Non- Homogeneous Partial Differential Equation

34. Solve $(D^2 - DD'+2D)z = e^{2x+y} + 4$ **[AU N/D 2012]**

Solution:

Given $(D^2 - DD'+2D)z = e^{2x+y} + 4$

$D(D - D'+2)z = e^{2x+y} + 4e^{0x+0y}$

$(D - 0D'-0)[D - D'-(-2)]z = e^{2x+y} + 4e^{0x+0y}$

By working rule,

If $(D - mD'-c)z = 0$, then $z = e^{cx} f(y + mx)$ where c is arbitrary.

$m_1 = 0, c_1 = 0$

$m_2 = 1, c_2 = -2$

$C.F = e^{0x} f_1(y + 0x) + e^{-2x} f_2(y + x)$

$= f_1(y) + e^{-2x} f_2(y + x)$

$$P.I_1 = \frac{1}{D^2 - DD'+2D} e^{2x+y}$$

$$= \frac{1}{4 - 2 + 4} e^{2x+y} = \frac{1}{6} e^{2x+y} \ (D \to 2, D' \to 1)$$

$$P.I_2 = \frac{1}{D^2 - DD'+2D} 4e^{0x+0y}$$

$$= x\frac{1}{2D - D'+2} 4e^{0x+0y} = x\frac{1}{2} 4e^{0x+0y} = 2x$$

Result: $\therefore z = C.F + P.I$

$$z = f_1(y) + e^{-2x} f_2(y + x) + 2x + \frac{1}{6} e^{2x+y}$$

35. Solve: $(D^2 + 2DD'+D'^2 - 2D - 2D')z = \sin(x + 2y)$ **[AU M/J 2016]**

Solution:

Given $(D^2 + 2DD'+D'^2 - 2D - 2D') = (D + D')(D + D' - 2)$

If $(D - m_1D'-\alpha_1)(D - m_2D'-\alpha_2)z = 0$ **then**

Formula : $z = e^{\alpha_1 x} f_1(\mathrm{y} + \mathrm{m}_1 x) + e^{\alpha_2 x} f_2(\mathrm{y} + \mathrm{m}_2 x)$

Here $m_1 = -1, \quad \alpha_1 = 0$

$m_2 = -1, \quad \alpha_2 = 2$

$\therefore \quad C.F = e^{0x} f_1(\mathrm{y} - 1x) + e^{2x} f_2(\mathrm{y} - 1x)$

$= f_1(\mathrm{y} - x) + e^{2x} f_2(\mathrm{y} - x)$

$$\mathrm{P.I}_1 = \frac{1}{(D + D')(D + D' - 2)} \sin(x + 2y)$$

$$= \frac{1}{(D + D')(D + D' - 2)} \ I.P\ e^{i(x+2y)}$$

$$= I.P \frac{1}{(D + D')(D + D' - 2)} \ e^{ix+i2y}$$

$$= I.P\frac{1}{(i+2i')(i+2i-2)} \quad e^{ix+i2y} \qquad \text{Re}\,place \;\; D \to i,\, D' \to 2i$$

$$= I.P\frac{1}{3i(3i-2)} \quad e^{ix+i2y}$$

$$= I.P\frac{-i}{3}\frac{(3i+2)}{(3i+2)(3i-2)} \quad e^{ix+i2y}$$

$$= I.P\frac{-i}{3}\frac{(3i+2)}{-9-4} \quad e^{ix+i2y}$$

$$= I.P\frac{-3+2i}{39} \quad e^{ix+i2y}$$

$$= I.P\frac{-3+2i}{39}[\cos(x+2y)+i\sin(x+2y)]$$

$$= \frac{-3}{39}\sin(x+2y)+\frac{2}{39}\cos(x+2y)$$

Result:

The general solution is

z = C.F + P.I

$$z = f_1(y-x)+e^{2x}f_2(y-x)\frac{-3}{39}\sin(x+2y)+\frac{2}{39}\cos(x+2y)$$

36. Solve $(D^2-3DD'+2D'^2+2D-2D')z = Sin(2x+y)$ **[AU A/M 2017]**

Solution:

Given $(D^2-3DD'+2D'^2+2D-2D')z = Sin(2x+y)$

To find C.F:

Take $(D-2D')(D-D'+2)z = 0$

By working rule,

If $(D-m_1D'-c_1)(D-m_2D'-c_2)z = 0$

Then $z = e^{c_1x}f_1(y+m_1x)+e^{c_2x}f_2(y+m_2x)$

Here $m_1 = 2 \quad c_1 = 0$

$m_2 = 1 \quad c_2 = -2$

$$C.F = e^{0x}f_1(y+2x)+e^{-2x}f_2(y+x)$$

$$\text{P.I} = \frac{1}{[(D^2-3DD'+2D'^2+2D-2D')]}\sin(2x+y) \qquad replace\,D^2by-2^2 = -4$$

$$\text{P.I} = \frac{1}{[(-4)-3(-2)+2(-1)+2D-2D')]}\sin(2x+y) \qquad DD'by-(2)(1) = -2$$

$$= \frac{1}{2D-2D'}\sin(2x+y) \qquad D'^2\,by-(1)^2 = -1$$

$$= \frac{(2D+2D')}{(2D-2D')(2D+2D')}\sin(2x+y)$$

$$=\frac{(2D(\sin(2x+y))+2D'(\sin(2x+y))}{(4D^2-4D'^2\ \ )}$$

$$=\frac{(2(Cos(2x+y)2)+2(Cos(2x+y))}{(4(-4)-4(-1))}$$

$$=\frac{6Cos(2x+y)}{-12}$$

Result:

z = C.F + P.I

$$z=f_1(y+2x)+e^{-2x}f_2(y+x)-\frac{1}{2}\cos(2x+y)$$

37. Solve $(2D^2-DD'-D'^2+6D+3D')z=xe^y$ **[AU N/D 2010]**

Solution:

Given $(2D^2-DD'-D'^2+6D+3D')z=xe^y$

$$\Rightarrow\left[(2D+D')(D-D')+3(2D+D')\right]z=xe^y$$

$$(2D+D')(D-D'+3)z=xe^y$$

$$\text{To find C.F}:(2D+D')(D-D'+3)z=0$$

$$\Rightarrow\left[D-\left(\frac{-1}{2}D'\right)\right]\left[D-D'-(-3)\right]z=0$$

$$here, m_1=1, c_1=-3, m_2=\frac{-1}{2}, c_2=0$$

$$C.F=e^{-3x}f_1(y+x)+f_2(y-\frac{1}{2}x)$$

$$P.I=\frac{1}{2D^2-DD-D'^2+6D+3D'}xe^y$$

$$=e^y\frac{1}{2D^2-D(D+1)-(D+1)^2+6D+3(D+1)}x \qquad \text{replace}D'\text{ by }D'+1$$

$$=e^y\frac{1}{2D^2-DD-D-D^2-2D-1+6D+3D+3}x$$

$$=e^y\frac{1}{2+5D+D'+2D^2-DD-D^2}x$$

$$=\frac{e^y}{2}\left[\frac{1}{1+\frac{1}{2}\left(5D+D'+2D^2-DD'-D'^2\right)}\right]x$$

$$=\frac{e^y}{2}\left[1+\frac{1}{2}\left(5D+D'+2D^2-DD'-D'^2\right)\right]^{-1}x$$

$$=\frac{e^y}{2}\left[1-\frac{1}{2}\left(5D+D'+2D^2-DD'-D'^2\right)\right]x$$

$$=\frac{e^y}{2}\left[x-\frac{1}{2}\big(5(1)+0+0\big)\right]=\frac{e^y}{2}\left[x-\frac{5}{2}\right]=\frac{e^y}{4}(2x-5)$$

Result: The general solution is z=C.F+P.I

$$z=e^{-3x}f_1(y+x)+f_2(y-\frac{1}{2}x)+\frac{e^y}{4}(2x-5)$$

38. Solve $(D^2-2DD'+D'^2-3D+2D'+2)z=e^{2x-y}$ **[AU N/D 2001]**

Solution:

Given $(D^2-2DD'+D'^2-3D+2D'+2)z=e^{2x-y}$

To find C.F:

Take $(D-D'-1)(D-D'-2)z=0$

By working rule,

If $(D-m_1D'-c_1)(D-m_2D'-c_2)z=0$

Then $z=e^{c_1z}f_1(y+m_1x)+e^{c_2z}f_2(y+m_2x)$

Here $m_1=1 \quad c_1=1$

$m_2=1 \quad c_2=2$

$$C.F=e^{x}f_1(y+x)+e^{2x}f_2(y+x)$$

$$P.I=\frac{1}{(D-D'-1)(D-D'-2)}e^{2x-y}$$

$$=\frac{1}{(2+1-1)(2+1-2)}e^{2x-y}=\frac{1}{2}e^{2x-y}$$

Result: $\therefore \quad z=C.F+PI$

$$z=e^{x}f_1(y+x)+e^{2x}f_2(y+x)+\frac{1}{2}e^{2x-y}$$

Formation of Partial Differential Equation

39. Form the PDE by eliminating the arbitrary constants ϕ from

$\phi(x^2+y^2+z^2,ax+by+cz)=0$

Solution: **[AU N/D 2010]**

Given $u=x^2+y^2+z^2$

$$\frac{\partial u}{\partial x}=2x+2z\frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x}=2x+2zp$$

$$\frac{\partial u}{\partial y}=2y+2z\frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y}=2y+2zq$$

$v=ax+by+cz$

$$\frac{\partial v}{\partial x}=a+c\frac{\partial z}{\partial x}$$

$$\frac{\partial v}{\partial x}=a+cp$$

$$\frac{\partial v}{\partial y}=b+c\frac{\partial z}{\partial y}$$

$$\frac{\partial v}{\partial y}=b+cq$$

Required PDE is given by,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x+2zp & a+cp \\ 2y+2zq & b+cq \end{vmatrix} = 0$$

$$(2x+2pz)(b+cq)-(a+cp)(2y+2qz)=0$$

$$2bx+2cqx+2pbz+2pczq-2ay-2aqz-2cpy-2cpqz=0$$

$$(2bz-2cy)p+(2cx-2az)q=2ay-2bx$$

$$(bz-cy)p+(cx-az)q=ay-bx$$

40. Form the PDE by eliminating the arbitrary function 'f' and 'g' from $z = x^2 f(y) + y^2 g(x)$.

Solution: **[AU N/D 2013]**

$$z = x^2 f(y) + y^2 g(x) \quad ...(1)$$

$$p = \frac{\partial z}{\partial x} = 2xf(y) + y^2 g'(x) \quad ...(2)$$

$$q = \frac{\partial z}{\partial y} = x^2 f'(y) + 2yg(x) \quad ...(3)$$

$$r = \frac{\partial z}{\partial x \partial y} = 2xf'(y) + 2yg'(x) \quad ...(4)$$

$$s = \frac{\partial^2 z}{\partial x^2} = 2f(y) + y^2 g''(x) \quad ....(5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = x^2 f''(y) + 2g(x) \quad ....(6)$$

$$(3) \Rightarrow f'(y) = \frac{q-2g(x)y}{x^2} \quad ....(7)$$

$$(2) \Rightarrow g'(x) = \frac{p-2f(y)x}{y^2} \quad ....(8)$$

Substituting (7) and (8) in (4),

$$r = 2x\left[\frac{q-2g(x)y}{x^2}\right] + 2y\left[\frac{p-2f(y)x}{y^2}\right]$$

$$= 2\left[\frac{y(q-2g(x)y)+x(p-2f(y)x)}{xy}\right]$$

$$xyr = 2\left[qy+px-2\left[y^2 g(x)+x^2 f(y)\right]\right]$$

$$xyr = 2[px+qy-2z]$$

41. Form a PDE eliminating the arbitrary function from the relation

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$$

Solution: **[AU N/D 2011, M/J 2007, 2014]**

Given: $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

P.d.w.r.to ‘ x ’ & ‘ y ’ we get,

$$p = \frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$= \frac{-2}{x^2} f'\left(\frac{1}{x} + \log y\right)$$

$$\frac{-px^2}{2} = f'\left(\frac{1}{x} + \log y\right) \qquad ....(1)$$

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$= 2y + \frac{2f'}{y}\left(\frac{1}{x} + \log y\right)$$

$$\frac{(q-2y)y}{2} = f'\left(\frac{1}{x} + \log y\right) \qquad ...(2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\left(\frac{-px^2}{2}\right)}{\frac{(q-2y)y}{2}} = 1$$

$$\frac{-px^2}{(q-2y)y} = 1$$

Result: $-px^2 = qy - 2y^2$

$$px^2 + qy = 2y^2 \qquad \textit{is the required p.d.e.}$$

42. Form the partial differential equation by eliminating f and ϕ form $z = xf\left(\frac{y}{x}\right) + y\phi(x)$

[AU N/D 2016]

Solution:

Given: $z = xf\left(\frac{y}{x}\right) + y\phi(x)$

Differentiate with respect to ‘x’ and ‘y’

$$p = xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\phi'(x)$$

$$p = -\frac{y}{x}f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) + y\phi'(x)....(1)$$

$$q = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) + \phi(x)$$

$$q = f'\left(\frac{y}{x}\right) + \phi(x)....(2)$$

Differentiate once again with respect to 'x' and 'y'

$$r = \frac{\partial^2 z}{\partial x^2} = ...$$

Differentiate (2) with respect to 'x'

$$s = \frac{\partial^2 z}{\partial x \partial y} = f''\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + \phi'(x)....(3)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right)\left(\frac{1}{x}\right).....(4)$$

$(1)\times x + (2)\times y$ gives

$$px + qy = -yf'\left(\frac{y}{x}\right) + xf\left(\frac{y}{x}\right) + xy\phi'(x) + yf'\left(\frac{y}{x}\right) + y\phi(x)$$

$$= xy\phi'(x) + xf\left(\frac{y}{x}\right) + y\phi(x)$$

$$px + qy = xy\phi'(x) + z.....(5)$$

use (4) in (3)

$$s = -\frac{y}{x}\times t + \phi'(x)$$

$$\frac{xs + yt}{x} = \phi'(x) \quad \text{use this } \phi'(x) \text{ in (5)}$$

$$px + qy = z + xy\left[\frac{xs + yt}{x}\right]$$

$$px + qy = z + xys + y^2 t$$

$$z = px + qy - xys - y^2 t \quad \text{is the required equation.}$$

Result: $z = px + qy - xys - y^2 t$

43. Find the Partial differential equation of all planes which are at a constant distance 'k' from the origin. [AU N/D 2016]

Solution:

The equation of the plane in normal form is $lx + my + nz = k \quad ...(1)$

Where l, m ,n are direction cosines of the normal form the origin to the plane.

Then $l^2+m^2+n^2=1\ (or)\ n=\sqrt{1-l^2-m^2}$

$\therefore$ (1) becomes

$$lx+my+\left(\sqrt{1-l^2-m^2}\right)z=k \quad ...(2)$$

Differentiating (2) partially with respect to ‘x’ and ‘y’ we get

$$l+\left(\sqrt{1-l^2-m^2}\right)p=0 \quad and \quad m+\left(\sqrt{1-l^2-m^2}\right)q=0...(3)$$

$$l=-\left(\sqrt{1-l^2-m^2}\right)p \quad and \quad m=-\left(\sqrt{1-l^2-m^2}\right)q...(4)$$

Let us consider, $l^2+m^2=\left(\sqrt{1-l^2-m^2}\right)\left(p^2+q^2\right)$

$$l^2+m^2=p^2+q^2-\left(l^2+m^2\right)\left(p^2+q^2\right)$$

$$\left(l^2+m^2\right)\left(1+p^2+q^2\right)=p^2+q^2$$

$$l^2+m^2=\frac{p^2+q^2}{1+p^2+q^2}$$

$$1-l^2-m^2=1-\frac{p^2+q^2}{1+p^2+q^2}$$

$$=\frac{1}{1+p^2+q^2}$$

From (4), $l=-\frac{p}{\sqrt{1+p^2+q^2}}$ and $m=-\frac{q}{\sqrt{1+p^2+q^2}}$

Substituting the values of $l,m\,and\ 1-l^2-m^2$ in (2) we get

$$-\frac{px}{\sqrt{1+p^2+q^2}}-\frac{qy}{\sqrt{1+p^2+q^2}}+\frac{1}{\sqrt{1+p^2+q^2}}z=k$$

$z=px+qy+a\sqrt{1+p^2+q^2}$ is the required PDE.

Result: $z=px+qy+a\sqrt{1+p^2+q^2}$

Non-Linear Partial Differential Equation

44. Solve $p^2+q^2=x^2+y^2$ **[AU N/D 2009 M/J 2003, 2005]**

Solution:

Given $p^2+q^2=x^2+y^2$

Type (4) = $F_1(x,p)=F_2(y,q)$

Let $p^2-x^2=y^2-q^2=a^2$

$p^2-x^2=a^2$, $p=\sqrt{x^2+a^2}$

$y^2-q^2=a^2$, $q=\sqrt{y^2-a^2}$

Sub (1) and (2) in (3) We know that $Z = \int p\,dx + \int q\,dy$

$$z = \int \sqrt{x^2 + a^2}\; dx + \int \sqrt{y^2 - a^2}\; dy$$

Result: $z = \dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{x}{a}\right) + \dfrac{x\sqrt{x^2 + a^2}}{2} + \dfrac{y\sqrt{y^2 - a^2}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{y}{a} + b$

Which is C.I there is no S.I

45. Solve $9pqz^4 = 4(1+z^3)$ **[AU M/J 2007, 2008]**

Solution:

Given $9pqz^4 = 4(1+z^3)$ ……… (1)

Type $F(z, p, q) = 0$ $9pqz^4 = 4(1+z^3)$

Let $Z = f(x + ay)$ be the solution of (1)

put $x + ay = U$

$$\frac{\partial u}{\partial x} = 1\ ; \quad \frac{\partial u}{\partial y} = a \qquad \text{……… (A)}$$

$z = f(u)$

$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}\cdot 1 \\ q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}\cdot a \end{aligned}\right\}\text{using (A)} \rightarrow (2)$$

Sub (2) in (1)we get

$$9\left(\frac{\partial z}{\partial u}\right)\left(a\frac{\partial z}{\partial u}\right)z^4 = 4(1+z^3)$$

$$\left(\frac{dz}{du}\right)^2 = \frac{4(1+z^3)}{9az^4}$$

$$= \frac{4}{9a}\;\frac{(1+z^3)}{z^4}$$

$$\therefore \frac{dz}{du} = \frac{2}{3\sqrt{a}}\sqrt{\frac{1+z^3}{z^4}}$$

$$\frac{\sqrt{a}}{2}\;\frac{3z^2dz}{\sqrt{1+z^3}} = \int du$$

put $1+z^3 = t$

$3z^2dz = dt$

$$\frac{\sqrt{a}}{2}\int\frac{dt}{\sqrt{t}} = u + b$$

$$\frac{\sqrt{a}}{2}\;2\sqrt{t} = x + ay + b$$

Result: $\sqrt{a}\,\sqrt{1+z^3} = x + ay + b \qquad (\because t = 1+z^3)$

Which is C.I

46. Solve $x^2p^2 + xpq = z^2$ **[AU M/J 2009,2010]**

Solution:

Given $x^2p^2 + xpq = z^2$

$$(xp)^2 + (xp)q = z^2 \qquad \ldots.. (1)$$

Type (5), $F(z, x^m, p, y^n q) = 0$

Here $m = 1$

put $X = \log x$

$$\frac{\partial X}{\partial x} = \frac{1}{x}$$

let $P = \dfrac{\partial X}{\partial x}$

Now $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial x}$

$$p = P\,\frac{1}{x}$$

$$xp = P \qquad \ldots. (2)$$

sub (2) in (1) we get

$$P^2 + Pq = z^2 \qquad \ldots. (3)$$

$f(z, p, q) = 0$

let $Z = f(X + ay)$

$u = X + ay$

$$\frac{\partial u}{\partial x} = 1\ ; \quad \frac{\partial u}{\partial y} = a$$

$$\left.\begin{aligned} P &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}\cdot \\ Q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}\cdot a \end{aligned}\right\} \qquad \ldots. (4)$$

Sub (4)in (3)

$$\left(\frac{dz}{du}\right)^2 + \frac{dz}{du}.a\,\frac{dz}{du} = z^2$$

$$\left(\frac{dz}{du}\right)^2 (1+a) = z^2$$

$$\frac{dz}{du} = \pm\frac{z}{\sqrt{(1+a)}}$$

$$\frac{dz}{z} = \pm\frac{du}{\sqrt{(1+a)}}$$

$$\int \frac{dz}{z} = \pm \int \frac{du}{\sqrt{(1+a)}}$$

$$\log z = \pm \frac{1}{\sqrt{(1+a)}}(u) + b = \pm \frac{1}{\sqrt{(1+a)}}(X + ay) + b \qquad [\therefore U = X + ay]$$

Result: $\log z = \pm \frac{1}{\sqrt{(1+a)}} (\log x + ay) + b$

Which is C.I

47. Solve $x^2p^2 + y^2q^2 = z^2$ **[AU M/J 2014]**

Solution:

Given $x^2p^2 + y^2q^2 = z^2$

$$(xp)^2 + (yq)^2 = z^2 \qquad \text{….. (1)}$$

Type (5), $F(z, x^m, p, y^n q) = 0$

Here $m = 1$, $n=1$

put $X = \log x$ Put $Y = \log y$

$$\frac{\partial X}{\partial x} = \frac{1}{x} \qquad \frac{\partial Y}{\partial y} = \frac{1}{y}$$

let $P = \frac{\partial z}{\partial X}$ $\quad Q = \frac{\partial z}{\partial Y}$

Now $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x}$ $\quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$

$$p = P\,\frac{1}{x} \qquad q = Q\,\frac{1}{y}$$

$$xp = P \qquad yq = Q$$

sub in (1) we get

$$P^2 + Q^2 = z^2 \qquad \text{…. (2)}$$

$f(z, p, q) = 0$

$u = X + aY$

$$\frac{\partial u}{\partial x} = 1\ ; \quad \frac{\partial u}{\partial y} = a$$

$$\left.\begin{aligned} P &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}\cdot \\ Q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}\cdot a \end{aligned}\right\} \qquad \text{…. (3)}$$

Sub (3)in (2)

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 (1+a^2) = z^2$$

$$\frac{dz}{du} = \pm\frac{z}{\sqrt{(1+a^2)}}$$

$$\frac{dz}{z} = \pm\frac{du}{\sqrt{(1+a^2)}}$$

$$\int\frac{dz}{z} = \pm\int\frac{du}{\sqrt{(1+a^2)}}$$

$$\log z = \pm\frac{1}{\sqrt{(1+a^2)}}(u) + b$$

$$= \pm\frac{1}{\sqrt{(1+a^2)}}(X+ay)+b \quad [\ \therefore U = X+ay]$$

Result: $\log z = \pm\frac{1}{\sqrt{(1+a^2)}}(\log x + a\log y) + b$

which is C.I

48. Find the singular integral if $z = px + qy + \sqrt{1+p^2+q^2}$

Solution: **[AU N/D 2010, 2013 M/J 2013]**

Given $z = px + qy + \sqrt{1+p^2+q^2}$

This is of the form $z = px + qy + f(p,q)$ (Clairaut's form)

Hence, the complete integral is $z = ax + by + \sqrt{1+a^2+b^2}$

Where a and b are arbitrary constants.

Singular solution is found as follows.

$$z = ax + by + \sqrt{1+a^2+b^2} \quad \ldots\ldots (1)$$

Diff (1) w.r.to a & b, we get

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}}$$

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$$

$$x = -\frac{a}{\sqrt{1+a^2+b^2}} \quad \ldots\ldots (2)$$

$$y = -\frac{b}{\sqrt{1+a^2+b^2}} \quad \ldots\ldots (3)$$

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2)=1-\frac{a^2+b^2}{1+a^2+b^2}$$

$$1-x^2-y^2=\frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$1-x^2-y^2=\frac{1}{1+a^2+b^2}$$

$$\sqrt{1-x^2-y^2}=\frac{1}{\sqrt{1+a^2+b^2}}$$

$$\sqrt{1-a^2-b^2}=\frac{1}{\sqrt{1-x^2-y^2}}$$

$(2)\,\&\,(3)$ *because*

$$x=-a\sqrt{1-x^2-y^2}$$

$$y=-b\sqrt{1-x^2-y^2}$$

$$a=\frac{-x}{\sqrt{1-x^2-y^2}},\ b=\frac{-y}{\sqrt{1-x^2-y^2}}$$

sub in (1), we get

$$\mathrm{z}=\frac{-\mathrm{x}^2}{\sqrt{1-\mathrm{x}^2-\mathrm{y}^2}}-\frac{-\mathrm{y}^2}{\sqrt{1-\mathrm{x}^2-\mathrm{y}^2}}+\frac{1}{\sqrt{1-\mathrm{x}^2-\mathrm{y}^2}}$$

$$=\frac{1-\mathrm{x}^2-\mathrm{y}^2}{\sqrt{1-\mathrm{x}^2-\mathrm{y}^2}}$$

$$\mathrm{z}=\sqrt{1-\mathrm{x}^2-\mathrm{y}^2}$$

$$z^2=1-\mathrm{x}^2-\mathrm{y}^2$$

$\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2=1$ *is the* singular *solution*

Note: *Put* $b=\phi(a)$ *in* (1)

$$z=ax+\phi(a)y+\sqrt{1+\mathrm{a}^2+[\phi(a)]^2} \qquad \text{…. (4)}$$

Diff (4) P.w.r.to 'a', we get

$$0=x+\phi'(a)y+\frac{[2a+2\phi(a)\phi'(a)]}{2\sqrt{\sqrt{1+\mathrm{a}^2+[\phi(a)]^2}}} \qquad \text{…. (5)}$$

Eliminate 'a' between (4) & (5), we get the general solution.

49. Find the general solution of $z=px+qy+p^2+pq+q^2$ **[AU A/M 2017, 18]**

Solution:

Given: $z=px+qy+p^2+pq+q^2$

This is of Clairraunt's form

The complete solution is $z=ax+by+a^2+ab+b^2$ .....(1)

Diff. partial with respect to 'a' and 'b'

$$\frac{\partial z}{\partial a} = x + 2a + b \Rightarrow 2a + b + x = 0$$
$$\frac{\partial z}{\partial b} = y + a + 2b \Rightarrow a + 2b + y = 0$$

Solving for 'a' and 'b'

$$\frac{a}{\begin{vmatrix} 1 & x \\ 2 & y \end{vmatrix}} = \frac{b}{\begin{vmatrix} x & 2 \\ y & 1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}$$
$$\frac{a}{y-2x} = \frac{b}{x-2y} = \frac{1}{4-1}$$
$$\frac{a}{y-2x} = \frac{b}{x-2y} = \frac{1}{3}$$

ie
$$\frac{a}{y-2x} = \frac{1}{3} \quad and \quad \frac{b}{x-2y} = \frac{1}{3}$$
$$a = \frac{y-2x}{3} \quad and \quad b = \frac{x-2y}{3}$$

Sub. in (1)

$$z = \frac{xy-2x^2}{3} + \frac{xy-2y^2}{3} + \frac{(y-2x)^2}{9} + \frac{(y-2x)(x-2y)}{9} + \frac{(x-2y)^2}{9}$$

$$z = \frac{1}{9}\left[3xy - 6x^2 + 3xy - 6y^2 + y^2 - 4xy + 4x^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy + 4y^2\right]$$
$$9z = 3xy - 3x^2 - 3y^2$$
$$3z = xy - x^2 - y^2$$

Which is a Singular Integral.

Result: $3z = xy - x^2 - y^2$

50. Solve $z^2(p^2+q^2)= x^2+y^2$ **[AU N/D 2011, '15]**

Solution:

Given $z^2(p^2+q^2) = x^2+y^2$

$$(zp)^2 + (zq)^2 = x^2 + y^2 \qquad \ldots. (1)$$

This equation is of the form $f_1(x, z^m p) f_2(y, z^m q)$ $(type\, 6)$

$here\, m \neq 1$

$$\therefore Z = z^{m+1}$$
$$Z = z^2$$
$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x}$$

$$p = 2zp \Rightarrow \frac{P}{2} = zp$$

Similarly $\frac{Q}{2} = zp$

substitutein equation(1), we get

$$\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = x^2 + y^2$$

$$P^2 + Q^2 = 4(x^2 + y^2)$$

$$P^2 - 4x^2 = 4y^2 - Q^2$$

This equation is of the form $f_1(x,P) = f_2(y,Q)$ $(type 4)$

$$\therefore P^2 - 4x^2 = 4y^2 - Q^2 = 4a^2 (say)$$

$$P^2 = 4x^2 + 4a^2, \quad Q = 4y^2 - 4a^2$$

$$p = 2\sqrt{x^2 + a^2}, \quad Q = 2\sqrt{y^2 - a^2}$$

$$dZ = Pdx + Qdy$$

$$dZ = 2\sqrt{x^2 + a^2}\,dx + 2\sqrt{y^2 - a^2}\,dy$$

$$\int dZ = 2\int \sqrt{x^2 + a^2}\,dx + 2\int \sqrt{y^2 - a^2}\,dy$$

$$Z = 2\left[\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a}\right] + b$$

Result: $z^2 = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1}\frac{x}{a} + y\sqrt{y^2 - a^2} - a^2 \cosh^{-1}\frac{y}{a} + b$

$$= x\sqrt{x^2 + a^2} + y\sqrt{y^2 - a^2} + a^2\left[\sinh^{-1}\frac{x}{a} - \cosh^{-1}\frac{y}{a}\right] + b$$

51. Solve $z = px + qy + p^2q^2$ and obtain its singular solution. **[AU A/M 2015]**

Sol:

$$z = px + qy + p^2q^2$$

This is of the type

$$z = px + qy + f(p,q)$$

The C.I is

$$z = ax + by + a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find singular integral:

$$(1)\ldots\ldots\ldots \frac{\partial z}{\partial a} = 0 \Rightarrow x + 2ab^2 = 0 \ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\partial z}{\partial b} = 0 \Rightarrow y + 2a^2b = 0 \ldots\ldots\ldots\ldots\ldots(3)$$

$$(2) \Rightarrow \quad x = -2ab^2 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$(3) \Rightarrow \quad y = -2a^2b \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$\frac{x}{b}=\frac{y}{a}=-2ab=\frac{1}{k}\ (say)$$

$$a=ky\ ,\ b=kx\ \ \ldots\ldots\ldots\ldots\ldots(6)$$

Put (6) in (4)

$$x=-2ky(kx)^2$$

$$=-2k^3yx^2$$

$$k^3=\frac{-1}{2xy}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Put (6) in (1)

$$z=kyx+kxy+(ky)^2(kx)^2$$

$$=2kxy+k^4x^2y^2$$

$$=2kxy+k\left(\frac{-1}{2xy}\right)x^2y^2 \qquad from \quad (7)$$

$$=2kxy-\frac{1}{2}kxy$$

$$=\frac{3}{2}kxy$$

$$z^3=\left(\frac{3}{2}kxy\right)^3$$

$$=\frac{27}{8}k^3x^3y^3$$

$$=\frac{27}{8}\left(\frac{-1}{2xy}\right)x^3y^3$$

$$=-\frac{27}{16}x^2y^2$$

Result: $16z^3+27x^2y^2=0$ *is the singular solution*

52. Find the singular integral of $z=px+qy+p^2-q^2$ **[AU N/D 2014,17]**

Solution:

$$z=px+qy+p^2-q^2$$

This is of the type

$$z=px+qy+f(p,q)$$

The C.I is

$$z=ax+by+a^2-b^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find singular integral:

$$\left.\begin{aligned}(1).........\frac{\partial z}{\partial a} &= 0 \Rightarrow x + 2a = 0 \Rightarrow a = \frac{-x}{2} \\ \frac{\partial z}{\partial b} &= 0 \Rightarrow y - 2b = 0 \Rightarrow b = \frac{-y}{2}\end{aligned}\right\}.....................(2)$$

Substituting (2) in (1), we get

$$z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{-x^2}{4} - \frac{y^2}{4}$$

$$= \frac{-x^2}{4} - \frac{y^2}{4}$$

Result: $y^2 - x^2 = 4z$ Which is the S.I

53. Obtain the complete solution of $p^2 + x^2y^2q^2 = x^2z^2$ **[AU A/M 2015]**

Solution:

Given $p^2 + x^2y^2q^2 = x^2z^2$

$\div x^2 \Rightarrow$

$$x^{-2}p^2 + y^2q^2 = z^2$$

$$\left(x^{-1}p\right)^2 + \left(yq\right)^2 = z^2 ....(1)$$

This is of the type $F\left(z, x^m p, y^n q\right) = 0$

Here $m = -1,\ n = 1$

Put $X = x^{1+1}$ $\quad Y = \log y$

$X = x^2$

$\dfrac{\partial X}{\partial x} = 2x$ $\quad \dfrac{\partial Y}{\partial y} = \dfrac{1}{y}$

Let $P = \dfrac{\partial z}{\partial X}$ $\quad Q = \dfrac{\partial z}{\partial Y}$

Now $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}\dfrac{\partial X}{\partial x}$ $\quad \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y}\dfrac{\partial Y}{\partial y}$

$p = P(2x)$ $\quad q = Q\dfrac{1}{y}$

$x^{-1}p = 2P.....................(2)$ $\quad yq = Q.....................(3)$

Sub. (2) and (3) in (1), we get

$$4P^2 + Q^2 = z^2 ....(4)$$

Let $z = f\left(X + aY\right)$ be the solution of (4)

Put $u = X + aY \Rightarrow \dfrac{\partial u}{\partial X} = 1, \dfrac{\partial u}{\partial Y} = a$

$\therefore\ z = f(u)$

$$P = \frac{\partial z}{\partial X} = \frac{dz}{du}\frac{\partial u}{\partial X}, \quad Q = \frac{\partial z}{\partial Y} = \frac{dz}{du}\frac{\partial u}{\partial Y}$$

$$P = \frac{dz}{du} \qquad Q = a\frac{dz}{du} \ldots\ldots\ldots\ldots\ldots(5)$$

Sub. (5) in (4)

$$4\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{4+a^2}$$

$$\left(\frac{dz}{du}\right) = \frac{z}{\sqrt{4+a^2}}$$

$$\frac{dz}{z} = \frac{du}{\sqrt{4+a^2}} \qquad \Rightarrow \int\frac{dz}{z} = \int\frac{du}{\sqrt{4+a^2}}$$

$$\log z = \frac{u}{\sqrt{4+a^2}} + b$$

$$\log z = \frac{X+aY}{\sqrt{4+a^2}} + b$$

Result: $\log z = \dfrac{x^2 + a\log y}{\sqrt{4+a^2}} + b$, Which is the Complete Integral.

ANNA UNIVERSITY QUESTIONS

1. Solve $x(y-z)p + y(z-x)q = z(x-y)$ [AU N/D 2011, 2014, A/M 2018] [Pg. no.19]
2. Solve $(mz - ny)p + (nx - lz)q = ly - mx$ [Pg. no.19]
3. Solve $(3z-4y)\dfrac{\partial z}{\partial x} + (4x-2z)\dfrac{\partial z}{\partial y} = 2y-3x$ [AU N/D 2008] [Pg. no.20]
4. Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ [AU N/D,,2015, 2016] [Pg. no.21]
5. Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$ [AU M/J 2013] [Pg. no.22]
6. Solve $(x^2 - y^2 - z^2)p + 2xyq - 2xz = 0$ [AU N/D 2001] [Pg. no.23]
7. Solve ($y - xz)p + (yz - x)q = (x + y)(x - y)$ [AU M/J 2005] [Pg. no.24]
8. Solve $pzx + qzy = y^2 - x^2$ [AU A/M 2009] [Pg. no.25]
9. Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$ [AU N/D 2016] [Pg. no.26]
10. Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$ [AU N/D 2010] [Pg. no.26]
11. Solve the partial differential equation $(x-2z)p + (2z-y)q = y-x$ [AU N/D 2017] [Pg. no.27]
12. Find the general solution of $x(y^2 - z^2) + y(z^2 - x^2)q = z(x^2 - y^2)$ [Pg. no.28]
13. Solve: $(y^2+z^2)p - xyq + xz = 0$ [AU N/D 2013] [Pg. no.29]

14. Solve the Lagrange's equation $(x+2z)p+(2xz-y)q=x^2+y.$ [AU M/J 2014] [Pg. no.30]
15. Find the general solution of $(z^2-y^2-2yz)p+(xy+zx)q=(xy-zx)$ [AU N/D 2015, A/M 2017] [Pg. no.31]
16. Solve $(D^2 - DD' - 30D'^2)y = xy + e^{6x+y}$ [AU M/J 2004, 2009] [Pg. no.31]
17. Solve $(D^2 - 6DD' - 5D'^2)z = e^x \sinh y + xy$ AU N/D 2006 , A/M 1997] [Pg. no.33]
18. Solve $(D^3 - 4D^2 D' + 4DD'^2)z = 6\sin(3x + 6y)$ [AU N/D 2010, A/M 2009] [Pg. no.34]
19. Solve $r+s-6t=y\cos x$ (or) $(D^2+DD'-6D'^2)Z=y\cos x$ AU M/J 2013, 2014, N/D 2016, A/M 2018] [Pg. no.35]
20. Solve $\left(D^2-5DD^1+6D^{12}\right)z=y\sin x$ [AU N/D 2017]] [Pg. no.36]
21. Solve $(D^3-2D^2D')z=2e^{2x}+3x^2y$ [AU N/D 2001] [Pg. no.37]
22. Solve the equation $\left(D^3+D^2D'-4DD'^2-4D'^3\right)z=\cos(2x+y)$ [AU N/D 2010] [Pg. no.38]
23. Solve the equation $(D^3-7DD'^2-6D'^3)z=\cos(x+2y)+x$ [AU N/D 2011,2012] [Pg. no.38]
24. Solve $\left(D^2-DD'-2D'^2\right)z=2x+3y+e^{2x+4y}$ [AU N/D 2013] [Pg. no.39]
25. Solve: $\left(D^2-2DD'\right)z=x^3y+e^{2x-y}$ [AU N/D 2014] [Pg. no.40]
26. Solve the equation $(D^3-7DD'^2-6D'^3)z=\sin(x+2y)$ [AU N/D 2011,2012,2014] [Pg. no.41]
27. Solve the equation $(D^2+4DD'-5D'^2)z=\sin(x-2y)+e^{2x-y}$ [AU A/M 2018] [Pg. no.42]
28.Solve: $(D^2-3DD'-2D'^2)z=(2+4y)e^{x+2y}$ [AU A/M 2015] [Pg. no.43]
29. Solve: $\left(D^3-2D^2D'\right)z=2e^{2x}+3x^2y$ [AU M/J 2016] [Pg. no.44]
30. Solve $\left(D^2+D^{1^2}\right)z=x^2y^2$ [AU N/D2015] [Pg. no.45]
31. Solve $\left(D^2+2DD^1+D^{1^2}\right)z=2\cos y-x\sin y$ [AU N/D2015] [Pg. no.46]
32. Solve $\left(D^2+2DD'+D'^2\right)z=x^2y+e^{x-y}$ [AU A/M 2017] [Pg. no.47]
33.Solve $\left(D^2+2DD'+D'^2\right)z=xy+e^{x-y}$ [AU N/D 2017] [Pg. no.49]
34. Solve $(D^2-DD'+2D)z=e^{2x+y}+4$ [AU N/D 2012] [Pg. no.50]
35. Solve: $(D^2+2DD'+D'^2-2D-2D')z=\sin(x+2y)$ [AU M/J 2016] [Pg. no.50]
36. Solve $(D^2-3DD'+2D'^2+2D-2D')z=Sin(2x+y)$ [AU A/M 2017] [Pg. no.51]
37. Solve $(2D^2-DD'-D'^2+6D+3D')z=xe^y$ [AU N/D 2010] [Pg. no.52]
38. Solve $(D^2-2DD'+D'^2-3D+2D'+2)z=e^{2x-y}$ [AU N/D 2001] [Pg. no.53]
39. Form the PDE by eliminating the arbitrary constants ϕ from $\phi(x^2+y^2+z^2, ax+by+cz)=0$ [AU N/D 2010] [Pg. no.54]

40. Form the PDE by eliminating the arbitrary function 'f' and 'g' from

$z = x^2 f(y) + y^2 g(x)$. [AU N/D 2013] [Pg. no.54]

41. Form a PDE eliminating the arbitrary function from the relation $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ [AU N/D 2011, M/J 2007, 2014] [Pg. no.55]
42. Form the partial differential equation by eliminating f and ϕ form $z = xf\left(\frac{y}{x}\right) + y\phi(x)$ [AU N/D 2016] [Pg. no.56].
43. Find the Partial differential equation of all planes which are at a constant distance 'k' from the origin. [AU N/D 2016] [Pg. no.57]
44. Solve $p^2 + q^2 = x^2 + y^2$ [AU N/D 2009 M/J 2003, 2005] [Pg. no.58]
45. Solve $9pqz^4 = 4(1+z^3)$ [AU M/J 2007, 2008] [Pg. no.58]
46. Solve $x^2p^2 + xpq = z^2$ [AU M/J 2009,2010] [Pg. no.59]
47. Solve $x^2p^2 + y^2q^2 = z^2$ [AU M/J 2014] [Pg. no.60]
48. Find the singular integral if $z = px + qy + \sqrt{1 + p^2 + q^2}$ [AU 2013 M/J 2013] [Pg. no.61]
49. Find the general solution of $z = px + qy + p^2 + pq + q^2$ [AU A/M 2017, 2018] [Pg. no.63]
50. Solve $z^2(p^2+q^2) = x^2+y^2$ [AU N/D 2011, '15][Pg. no: 64]
51. Solve $z = px + qy + p^2q^2$ and obtain its singular solution. AU A/M 2015] [Pg. no.65]
52. Find the singular solution of $z = px + qy + p^2 - q^2$ [AU N/D 2014,2017] [Pg. no.66]
53. Obtain the complete solution of $p^2 + x^2y^2q^2 = x^2z^2$ [AU A/M 2015] [Pg. no.66]

PARTIAL DIFFERENTIAL EQUATIONS
PART – A (2-Marks Questions)

1. Form a PDE by eliminating the arbitrary constants from $z = (x+a)^2 + (y-b)^2$ [Mar/Apr 2009]]
2. Form a PDE by eliminating a and b form $z = (x^2 + a^2)(y^2 + b^2)$ [Nov 2009]
3. Form a PDE by eliminating the arbitrary constants in $z = (x-a)^2 + (y-b)^2 + 1$ [Apr 2011]
4. Form a partial differential equation by eliminating the constants a and b form $z = ax^n + by^n$
5. Form a partial differential equation by eliminating the constants a and b form $z = ax^3 + by^3$ [AU M/J 2014]
6. Form a PDE by eliminating the arbitrary constants from $z = (2x^2 + a)(3y - b)$
7. Form a PDE by eliminating the arbitrary constants in $(x-a)^2 + (y-b)^2 + z^2 = 1$ Apr 2011,
8. Find the PDE of the family of sphere having their center on the line $x = y = z$ [AU Nov 2004]

9. Obtain the PDE by eliminating the arbitrary constant a and b from $z = xy + y\sqrt{x^2 - a^2} + b$
[AU Nov 2011, Apr 2007]
10. Form a PDE by eliminating the arbitrary function from $z = f(x^2 - y^2)$ [AU Apr 2008,]
11. Form a P.D.E by eliminating the arbitrary function from $z = f(x^2 + y^2)$ [AU Apr 2008]
12. Form a PDE by eliminating the arbitrary function ϕ from $\phi(x - y, x + y + z) = 0$
[AU Nov 2005, May 2004]
13. Eliminating the arbitrary function f from $z = f\left(\frac{xy}{z}\right)$ and form the PDE. [Apr 2007]
14. Solve $\frac{\partial z}{\partial x} = \sin x$ [May 2008, Apr 2007]
15. PDE by Form a eliminating the arbitrary function f from the relation
$z = x^2 + 2g\left(\frac{1}{y} + \log x\right)$
16.Solve $\sqrt{p} + \sqrt{q} = 1$ [May 2010, Mar 2008]
17.Find the singular integral of $p - q = \log(x + y)$ [Oct 95]
18. Solve $z = px + qy + 2\sqrt{pq}$ [Mar 2007, Dec 2004]
19. Solve the equation $yp = 2yx + \log q$ [Apr 2005, Nov 2001]
20. Solve the equation $p \tan x + q \tan y = \tan z$
21. Find the solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ [May 2010/ Mar 2007/2006]
22. Find the solution of $4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ [Apr 2006, May 2005]
23. What is the C.F of $(D^2 - 4DD' + 4D^2)z = x + y$ [Oct 98]
24. Write the particular integral of $(D^2 - DD')z = \sin(x + y)$
25. Solve $(D^2 + DD' - 2D'^2)z = 0$ [Nov 2009, May2007]
26. If the auxiliary equation $f(m) = 0$ has r equal roots $m_1 = m_2 = m_3 = \cdots = m_r$
27. what is the corresponding part in the complementary function?
28. Find the P.I of $(D^2 + 2DD' + D'^2)z = e^{x-y}$ [BDN, Apr 97]
29.Find the PDE of the family of spheres having their centers on the z-axis.
30. Solve the equation $(D - D')^3 z = 0$
31. Form the PDE by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$

32. Solve $(D^2 - 7DD' + 6D'^2)z = 0$
33. Find the complete integral of $p = 2qx$
34. Solve $(D-1)(D-D'+1)z = 0$ [Nov/Dec 2012]
35. Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 0$ [AU N/D 2013]
36. Solve $(D^4 - D'^4)z = 0$ [AU M/J 2014]
37. Define Clairaut's form
38. When PDE is said to be Linear
39.Distinguish between homogenous and non-homogenous p.d.e
40.Solve p cot x + q cot y = cot z

<u>PART-B</u>

1. (a)Solve $x(y-z)p + y(z-x)q = z(x-y)$ [Dec 2011, Apr 2008, May 2018]
   (b)Solve $(mz - ny)p + (nx - lz)q = ly - mx$ [Nov 86 / Apr 86]
2.(a) Solve $(3z - 4y)\frac{\partial z}{\partial x} + (4x - 2z)\frac{\partial z}{\partial y} = 2y - 3x$ [Dec 2008 / May 2000]
(b) Solve the partial differential equation $(x-2z)p + (2z-y)q = y - x$
[AU N/D 2017]
3.(a)Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$ [Nov 2008 /May/June 2013]
   (b).Solve $(x^2 - y^2 - z^2)p + 2xyq - 2xz = 0$
4. (a).Solve $(y - xz)p + (yz - x)q = (x + y)(x - y)$ [May 2005,Apr 2004]
   (b)Solve $pzx + qzy = y^2 - x^2$ [Dec 2008,Apr 2009]
5. (a) Solve $\left(D^2 + 2DD' + D'^2\right)z = xy + e^{x-y}$ [AU N/D 2017]
(b). Solve $\left(D^2 - 5DD^1 + 6D^{12}\right)z = y\sin x$ [AU N/D 2017]
6. (a).Solve $(D^3 - 4D^2 D' + 4DD'^2)z = 6\sin(3x + 6y)$ [Dec 2010,May 2009]
   (b).Solve $r + s - 6t = y\cos x$ [AU M/J 2013, 2014, 2018]
7. (a). Solve the equation $(D^2 + 4DD' - 5D'^2)z = \sin(x-2y) + e^{2x-y}$ [AU A/M 2018]
   (b). Form the PDE by eleminatin g the arbitary function g from relating
   $g\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$ [Mar 2007/Apr 2001]

8 . (a).Solve $p^2 + q^2 = x^2 + y^2$ [Dec 2009 / May 2005, 2003]

(b).Solve $9pqz^4 = 4(1+z^3)$ [May 2008,June 2007,Mar 2003]

9. (a).Solve $x^2p^2 + xpq = z^2$ [Apr 2010 / 2009]

(b).Solve $x^2p^2 + y^2q^2 = z^2$ [AU M/J 2014]

10. (a)Solve $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$ [Nov/Dec 2001]

(b).Solve $(D^2 - 2DD' + D'^2 - 3D + 2D' + 2)z = e^{2x-y}$ [Nov/Dec 2001]

11.(a) Find the singular integral if $z = px + qy + \sqrt{1+p^2+q^2}$ [M/J 2013,N/D 2013]

(b).Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$ [Nov/Dec 2010]

12.(a).Form the PDE by eliminating the arbitrary constants ϕ from

$\phi(x^2 + y^2 + z^2, ax + by + cz) = 0$ [Nov/Dec 2010]

(b).Solve $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y$ [Nov/Dec 2010]

13.(a)Solve the equation $\left(D^3 + 3^2D' - 4DD'^2 - 4D'^3\right)z = \cos(2x+y)$ [Nov/Dec 2010]

(b). Find the general solution of $z = px + qy + p^2 + pq + q^2$ [AU A/M 2017, 2018]

14.(a).Solve the equation $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x+2y) + x$

(b).Solve the partial differential equation $(x-2z)p + (2z-y)q = y - x$ [N/D 2011/2012]

15. (a).Solve $(D^2 - DD' + 2D)z = e^{2x+y} + 4$ [Nov/Dec 2012]

(b).Find the general solution of $x(y^2 - z^2) + y(z^2 - x^2)q = z(x^2 - y^2)$ [Apr 2001]

16.(a)Solve $\left(D^2 - DD' - 2D'^2\right)z = 2x + 3y + e^{2x+4y}$ [AU N/D 2013]

(b)Solve: $(y^2+z^2)p - xyq + xz = 0$ [AU N/D 2013]

17.(a). Form the PDE by eliminating the arbitrary function 'f' and 'g' from

Z = x^2 f(y) + y^2g(x). [AU N/D 2013]

(b).PDE by Form a eliminating the arbitrary function from the relation

$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ [Nov 2011, May 2007, M/J 2014]

18. a). Solve the Lagrange's equation $(x+2z)p+(2xz-y)q=x^2+y.$ [AU M/J 2014]

 b). Solve $(D - D' - 1)(D - D' - 2)\, z = e^{2x-y}$

19. a) . Solve $(2DD' + D'^2 - 3D')\, z = 3\cos(3x - 2y)$

 b). Solve $x^2p + y^2q = z$

20. a).Solve $z^2(p^2+q^2) = x^2+y^2$

 b). Solve $(zp + x)^2 + (zq + y)^2 = 1$

21.a) Solve $z = px+qy+p^2q^2$ and obtain its singular solution. [AU A/M 2015]

 b)Find the singular solution of $z = px+qy+p^2-q^2$ [AU N/D 2014, 2017]

MAILAM ENGINEERING COLLEGE

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II Year B.E (Civil, EEE & Mech.)

DEPARTMENT OF MATHEMATICS

SUBJECT NAME : MA8353 - TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS

UNIT – II (FOURIER SERIES)

Syllabus:

Dirichlet's Condition - General Fourier Series - Odd and Even functions – Half-range Sine series – Half-range Cosine series – Complex form of Fourier series – Parseval's Identity – Harmonic Analysis.

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Year	Nov-2017		April-2018	
	Q. No.	Pg. No.	Q. No.	Pg. No.
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	45	15	14	6
Part - B	8	29	9	31
	35	62	14	38
	15	40	35	62
	23	49	-	-

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PRINCIPAL

PART-A

1. State the sufficient condition for $f(x)$ to be expressed as a Fourier series.

Solution: **[A.U A/M 2017]**

Any function $f(x)$ can be developed as a Fourier series

$$\frac{a_o}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

where a_o , a_n , b_n are constants provided it satisfies the following Dirichlet's conditions

(i) $f(x)$ is periodic, single valued and finite.

(ii) $f(x)$ has a finite number of finite discontinuities in any one period and has no infinite discontinuity.

(iii) $f(x)$ has at the most a finite number of maxima and minima.

2. Obtain the first term of the Fourier series for the function $f(x)=x^2\,,\ -\pi<x<\pi$

Solution: **[A.U NOV. 2009]**

Given $f(x)=x^2\,,\ -\pi<x<\pi$

Here $f(-x)=(-x)^2$, therefore $f(x)$ is an even function.

Hence $b_n=0$

So we get $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$

First term of the Fourier series is $=\dfrac{a_0}{2}+a_1\cos x$ ....(1)

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2dx$$

$$=\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$a_0=\frac{2}{3}\pi^2$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$a_1=\frac{2}{\pi}\int_0^{\pi}x^2\cos x\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2d(\sin x)$$

$$=\frac{2}{\pi}\left\{x^2\left(\sin x\right)-2x\left(-\cos x\right)+2\left(-\sin x\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{2\pi(-1)\right\}$$

$$a_1 = -4$$

$$sub.in\ (1) \quad f(x) = \frac{\frac{2}{3}\pi^2}{2} + (-4)\cos x$$

$$= \frac{\pi^2}{3} - 4\cos x$$

3. If $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$**, deduce that** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$. **[A.U.MAY 2010]**

Solution:

Deduction : Put $x = \pi$ is a point of continuity

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n(-1)^n$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right\}$$

$$\frac{\pi^2}{6} = \left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right\}$$

$$(i.e.,)\ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$$

4. Find the constant term in the expansion of $\cos^2 x$ **as a Fourier series in the interval** $(-\pi, \pi)$.

Solution: **[A.U.NOV. 2010]**

Given $f(x) = \cos^2 x = \frac{1+\cos 2x}{2}$ is an even function in the interval $(-\pi, \pi)$

To find a_o :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{1+\cos 2x}{2}dx\ ,$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(1+\cos 2x)dx = \frac{1}{\pi}\left(x + \frac{\sin 2x}{2}\right)_0^{\pi} = 1$$

5. Find the Root mean square value of $f(x)=x^2$ in $(0\ ,\ l\ )$ **[A.U.NOV. 2010]**

Solution: **R.M.S** value $=\sqrt{\dfrac{\int_a^b [\,f(x)]^2\,dx}{b-a}}$

$$=\sqrt{\frac{\int_0^l (x^2)^2\,dx}{l-0}}$$

$$=\sqrt{\frac{\int_0^l x^4\,dx}{l}}$$

$$=\sqrt{\frac{\left(\frac{x^5}{5}\right)_0^l}{l}}$$

$$=\frac{l^2}{\sqrt{5}}$$

6. Give the expression for the Fourier series coefficient b_n **for the function** $f(x)$ **defined in** $(-2,2)$.

Solution: **[A.U MAY 2011]**

Here $2l=4,\ l=2$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$=\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

7. Without finding the values of a_0, a_n, b_n **the Fourier coefficients of Fourier series, for the function** $f(x)=x^2$ **in the interval** $(0,\pi)$ **Find the value of** $\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$.

Solution: **[A.U MAY 2011]**

We know that $\frac{1}{\pi}\int_0^{\pi}\left(f(x)\right)^2 dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$ ……………….(1)

$$(1)\text{x}2\Rightarrow\ \frac{2}{\pi}\int_0^{\pi}\left(f(x)\right)^2 dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

$$\frac{2}{\pi}\int_0^{\pi}x^4dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

$$\frac{2}{\pi}\left(\frac{x^5}{5}\right)_0^{\pi} = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

$$\frac{2}{5}\pi^4 = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

8. Find the RMS value of the function $f(x)=x$ in $(0,l)$. **[A.U.DEC 2011]**

Solution:

$$\textbf{R.M.S}\text{ value } = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

$$= \sqrt{\frac{\int_0^l (x)^2\,dx}{l-0}}$$

$$= \sqrt{\frac{\int_0^l x^2\,dx}{l}}$$

$$= \sqrt{\frac{\left(\frac{x^3}{3}\right)_0^l}{l}}$$

$$= \frac{l}{\sqrt{3}}$$

9. Define RMS value of a function $f(x)$ over the interval (a,b). **[A.U A/M 2018]**

Solution:

Let $f(x)$ be a function defined in an interval (a,b) then $\sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$ is called the RMS value.

10. Determine the value of a_n in the Fourier series expansion of $f(x)=x^3$ in $(-\pi,\pi)$.

Solution: **[A.U MAY 2008]**

Let $f(x)=x^3$

$f(-x)=-x^3$

$=-f(x)$

Therefore $f(x)$ is an odd function

Hence $a_0=0$ *and* $a_n=0$

11. If $f(x)=2x$ in the interval $(0,4)$, then find the value of a_2 in the Fourier series expansion .

Solution: **[A.U NOV.2008]**

Here $2l=4,\ l=2$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{1}{2}\int_0^{4} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$a_2 = \frac{1}{2}\int_0^{4} 2x\cos\frac{2\pi x}{2}\,dx$$

$$= \int_0^4 x\cos\pi x\,dx$$

$$= \left\{ x\left(\frac{\sin\pi x}{\pi}\right) + \frac{\cos\pi x}{\pi^2} \right\}_0^4$$

$$= 0$$

12. The Fourier series expansion of $f(x)$ in $(0,2\pi)$ is $f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}$. Find the RMS value of $f(x)$ in the interval $(0,2\pi)$. **[A.U NOV.2008]**

Solution: $f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}$

From this we get $a_0 = 0, a_n = 0, b_n = \frac{1}{n}$

RMS of $f(x)$ *in* $(0,2\pi)$ *is*

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2$$

13. Let $f(x)$ be defined in the interval $(0,2\pi)$ by $f(x) = \begin{cases} \dfrac{1+\cos x}{\pi - x}, & 0 < x < \pi \\ \cos x, & \pi < x < 2\pi \end{cases}$

$f(x+2\pi) = f(x)$. **Find the value of $f(\pi)$.** **[A.U MAY 2009]**

Solution: Is the discontinuous point in the middle .

$$f(x) = \frac{f(\pi-) + f(\pi+)}{2} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\lim_{x\to\pi}\frac{1+\cos x}{\pi - x} = \frac{1+\cos\pi}{\pi-\pi} = \frac{0}{0}$$

By using L Hospital rule

$$\lim_{x\to\pi}\frac{-\sin x}{-1} = \lim_{x\to\pi}\sin x = 0$$

$$(1) \Rightarrow f(x) = \frac{0+\cos\pi}{2} = \frac{-1}{2}$$

14. Find b_n in the expansion of x^2 as a Fourier series in $(-\pi,\pi)$ **[A.U A/M 2018]**

Solution:

Given $f(x)=x^2$ is an even function in the interval $(-\pi,\pi)$ $\therefore b_n=0$

15. If $f(x)$ is an odd function defined in $(-l,l)$,what are the values of a_0 , a_n.

Solution: **[AU Dec 2005]**

Given $f(x)$ **is an odd function in the interval** $(-l,l)$

$\therefore\ a_0=0,\ a_n=0$

16. In the Fourier series expansion of

$$f(x)=\begin{cases}1+\dfrac{2x}{\pi}, & -\pi<x<0\\ 1-\dfrac{2x}{\pi}, & 0<x<\pi\end{cases}\quad in\ (-\pi,\pi)$$ **Find the value of b_n.** **[A.U MAY 2005]**

Solution: Given $f(x)=\begin{cases}1+\dfrac{2x}{\pi}, & -\pi<x<0\\ 1-\dfrac{2x}{\pi}, & 0<x<\pi\end{cases}$

Let $f(x)=\begin{cases}\phi_1(x), & -\pi<x<0\\ \phi_2(x), & 0<x<\pi\end{cases}$

Where $\phi_1(x)=1+\dfrac{2x}{\pi},\ \phi_2(x)=1-\dfrac{2x}{\pi}$

Here $\phi_2(-x)=1+\dfrac{2x}{\pi}=\phi_1(x)$

The function $f(x)$ is an even function. Hence $b_n=0$.

17. Find the half range sine series for $f(x)=k\ in\ 0<x<\pi$. **[A.U MAY 2001]**

Solution: The sine series of $f(x)$ in $(0,\pi)$ is given by

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx\text{.........................}(1)$$

Where $b_n=\dfrac{2}{\pi}\int_0^{\pi} k\sin nx\,dx$

$$b_n=\frac{2k}{\pi}\left\{\frac{-\cos nx}{n}\right\}_0^{\pi}$$

$$=-\frac{2k}{n\pi}\{\cos n\pi-\cos 0\}\quad =\frac{2k}{n\pi}\{1-(-1)^n\}$$

$$b_n=\begin{cases}0 & when\ n\ is\ even\\ \dfrac{4k}{n\pi} & when\ n\ is\ odd\end{cases}$$

$$f(x)=\sum_{n\ is\ odd}^{\infty}\frac{4k}{n\pi}\sin nx\qquad f(x)=\frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}$$

18. If $f(x) = 3x - 4x^3 \; in \; -2 < x < 2$ then find the value of a_1. **[A.U DEC. 2009]**

Solution:

In the Fourier series expansion

$$f(x) = 3x - 4x^3 \; in \; -2 < x < 2$$

$$f(-x) = 3(-x) - 4(-x)^3$$

$$f(-x) = -\left[3x - 4x^3\right]$$

$$= -f(x)$$

Therefore given function is odd function.

Hence $a_0 = 0,\; a_n = 0$

Therefore $a_1 = 0$.

19. Write down the complex form of the Fourier series for $f(x) \; in \; (-l, l)$.

Solution: **[A.U N/D 2017]**

The complex form of the Fourier series for $f(x) \; in \; (-l, l)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

Where $C_n = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}} dx \qquad for \; all \; n = 0, \pm1, \pm2, ...$

20. State the Parseval's theorem in Fourier series. **[A.U MAY 2011]**

Solution:

Let $f(x)$ be a periodic function with period 2π defined in the interval $(-\pi, \pi)$.

Then

$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \left[f(x)\right]^2 dx = \dfrac{a_o^2}{4} + \dfrac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$ Where $a_o, a_n \; and \; b_n$ are Fourier coefficients of f(x)

21. If the Fourier series of $f(x) = \begin{cases} 0 & for \; -\pi \le x \le 0 \\ \sin x & for \; 0 \le x \le \pi \end{cases}$ is

$f(x) = \dfrac{1}{\pi} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left(\cos\frac{2nx}{4n^2 - 1} + \frac{1}{2}\sin x\right)$. **Find b_n.** **[A.U MAY 2011]**

Solution:

We know that $f(x) = \dfrac{a_o}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$

Given: $f(x) = \dfrac{1}{\pi} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\cos\frac{2nx}{4n^2 - 1} + \frac{1}{2}\sin x\right]$

Comparing we get $b_n = 0 \; (if \; n \neq 1)$

$$b_1 = -\frac{1}{\pi}$$

22. If $f(x)$ is expanded as a half range cosine series, Express $\int_0^l (f(x))^2 dx$ in terms of a_0 *and* a_n.

Solution: $$\frac{1}{l}\int_o^l [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

23. Find the RMS value of $f(x) = 1 - x$ *in* $0 < x < 1$ [A.U MAY 2010]

Solution:

Given $f(x) = 1 - x$ *in* $0 < x < 1$

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2 dx$$

$$\bar{y}^2 = \frac{1}{1-0}\int_0^1 [1-x]^2 dx$$

$$\bar{y}^2 = \int_0^1 [1 - 2x + x^2] dx$$

$$= \left\{ x - \frac{2x^2}{2} + \frac{x^3}{3} \right\}_0^1$$

$$\bar{y}^2 = \frac{1}{3}$$

$$\bar{y} = \frac{1}{\sqrt{3}}$$

24. If $f(x)$ is an odd function in the interval $(-1,1)$, what are the values of a_0 *and* a_n.

Solution: **[A.U MAY 2010]**

$\because$ $f(x)$ *is an odd function*

$a_0 = 0$ *and* $a_n = 0$.

25. Check whether $f(x) = 1 + x + x^2$ is odd or even.

Solution:

$$f(x) = 1 + x + x^2$$
$$f(-x) = 1 - x + (-x)^2$$
$$f(x) = 1 - x + x^2$$
$$f(-x) \neq -f(x)$$

$\therefore$ It is neither even nor odd.

26. Find the coefficient a_8 of $\cos 8x$ in the F.C.T of the function $f(x) = \sin 8x$ in $(0,\pi)$.

Solution:

Given $f(x) = \sin 8x$ is an odd function.

$\therefore a_n = 0$

$a_8 = 0$ Since it is odd function.

27. Determine b_n in Fourier series expansion $f(x)=\frac{1}{2}(\pi - x)$ in the interval $0<x<2\pi$.

Solution:

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\sin nx dx$$

$$= \frac{1}{2\pi}\left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\frac{\pi \cos 2n\pi}{n} + \frac{\pi \cos 0}{n}\right\}$$

$$= \frac{\pi}{2n\pi}\{1+1\}$$

$$b_n = \frac{1}{n}$$

28. If $x = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \dots\dots\dots\right\}$ in $0<x<\pi$. Prove that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution:

The sine series of $f(x)=x$ is given by $f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

The Parseval's Identity

$$\sum b_n^2 = \frac{2}{\pi}\int_0^{\pi}[f(\mathrm{x})]^2 dx$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

29. Find a_n in expanding e^{-x} as Fourier series $(-\pi,\pi)$.

Solution: Given $f(x)=e^{-x}$

$f(-x) = e^{-(-x)} = e^{x}$

$\therefore$ $f(x)$ is neither even nor odd function

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2+n^2}\left[-\cos nx + n\sin nx\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{-\pi}}{1+n^2}\left[-(-1)^n+0\right]\right\} - \left\{\frac{e^{\pi}}{1+n^2}\left[-(-1)^n+0\right]\right\}\right]$$

$$= \frac{1}{\pi}\left\{\frac{-(-1)^n e^{-\pi}}{1+n^2} + \frac{(-1)^n e^{\pi}}{1+n^2}\right\}$$

$$= \frac{(-1)^n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right] \qquad = \frac{(-1)^n}{\pi(1+n^2)}\left[2\sinh\pi\right]$$

30. Find the value of a_0 in the fourier series expansion of f(x) = e^x in $(0,2\pi)$.

Solution: **[AU Nov 2013]**

Given $f(x) = e^x$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x dx$$

$$= \frac{1}{\pi}\left\{e^x\right\}_0^{2\pi} = \frac{1}{\pi}\left[e^{2\pi} - 1\right]$$

31. What is the sum of the Fourier series at a point $x = x_o$ where the function $f(x)$ has a finite discontinuity.

Solution:

Let $f(x)$ can be expanded has a Fourier series

$$\frac{a_o}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \quad in \ \ (c, c+2l)$$

In the series converges to

$f(x_o)$ *if* x_o *is a point of continuity*

$\dfrac{f(x_o+0) + f(x_o-0)}{2}$ *if* $'x_o'$ *is a point of discontinuity*

32. In the expansion of $f(x) = \sinh x$ *in* $(-\pi, \pi)$ as a Fourier series, find the coefficient of b_n.

Solution:

$f(x) = \sinh x$, $f(-x) = \sinh(-x) = \sinh x = f(x)$

Since $f(x) \to$ even function, the Fourier coefficient $b_n = 0$.

33. Find the Fourier constants b_n for $x\sin x$ in $(-\pi, \pi)$. **[A.U DEC 2013]**

Solution:

Given $f(x) = x\sin x$ in $(-\pi, \pi)$

$$f(-x) = -x\sin(-x)$$

$$= x\sin x$$

$f(-x) = f(x)$

$f(x)$ is an even function

Hence $b_n = 0$

34. State Parseval's Identity for the half range cosine expansion of $f(x)$ in $(0,1)$.

Solution: $$\int_0^1 [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

35. State whether $y = \tan x$ can be expanded as a Fourier series. If so how ?If not why?

Solution:

Tan x cannot be expanded as a Fourier series. Since $\tan x$ does not satisfies Dirichlet's condition.

36. Write a_0, a_n in the expansion of $x + x^3$ as a Fourier series in $(-\pi, \pi)$. [A.U DEC 2010]

Solution:

$$f(x) = x + x^3$$

$$f(-x) = -x - x^3$$

$$= -(x + x^3)$$

$$= -f(x)$$

$\therefore$ $f(x)$ is an odd function in $(-\pi, \pi)$

Hence $a_0 = 0$ *and* $a_n = 0$

37. If $f(x) = x + x^2$ is expressed as a Fourier series in the interval $(-2,2)$ to which the value this series converges at $x = 2$. [A.U MAY2003]

Solution:

$x = 2$ is a point of discontinuity in the extreme.

$$\therefore \begin{matrix} f(x) \\ at\ x = 2 \end{matrix} = \frac{f(-2) + f(2)}{2}$$

$$= \frac{[(-2)^2 + (-2)] + [2^2 + 2]}{2}$$

$$= \frac{[4-2] + [4+2]}{2}$$

$$= 4$$

38. Find the sine series of function $f(x) = 1, \quad 0 \le x \le \pi$. [A.U N/D 2015, N/D 2016]

Solution:

The Sine series of $f(x)$ in $(0, \pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{Where } b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= \frac{2}{\pi}\left\{\frac{-\cos nx}{n}\right\}_0^{\pi}$$

$$= \frac{-2}{n\pi}\{\cos nx\}_0^{\pi}$$

$$= -\frac{2}{n\pi}\left\{(-1)^n - 1\right\}$$

$$= \frac{2}{n\pi}\left\{1-(-1)^n\right\}$$

$$b_n = \begin{cases} 0 \ if \quad n \ is \ even \\ \dfrac{4}{n\pi} \quad if \ \ n \ is \ odd \end{cases}$$

$$\therefore \ f(x) = \sum_{n=odd}^{\infty} \frac{4}{n\pi} \sin nx$$

$$f(x) = \frac{4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n} \sin nx$$

$$f(x) = \frac{4}{\pi}\sum_{n=1,2,3,...}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

39. Find the half range sine series expansion of f(x) = 1 in (0, 2) **[A.U Nov 2013]**

Solution:

In Sine series of $f(x)$ in $(0,2)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Here $l = 2$

$$\text{Where } b_n = \frac{2}{l}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$

$$= \frac{2}{2}\left\{\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right\}_0^2$$

$$= \frac{-2}{n\pi}\left\{\cos\frac{n\pi x}{2}\right\}_0^2$$

$$= -\frac{2}{n\pi}\left\{\cos\frac{2n\pi}{2} - 1\right\} = \frac{-2}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right)$$

40. What do you mean by Harmonic analysis? **[A.U MAY 2010/MAY2013]**

Solution:

The process of finding Euler Constants for a tabular function is known as Harmonic analysis.

41. If $f(x)=\begin{cases}\cos x & if\ \ 0<x<\pi \\ 50 & if\ \ \pi<x<2\pi\end{cases}$ **and** $f(x)=f(x+2\pi)\ \ for\ all\ \ x$ **Find the sum of the Fourier series of** $f(x)\ \ at\ \ x=\pi$ **.** **[A.U NOV 2007]**

Solution:

To Find $f(x)\ \ at\ x=\pi$ **:**

$x=\pi$ is a discontinuous point in the middle

$$f(\pi)=\frac{f(\pi-)+f(\pi+)}{2}\ldots\ldots\ldots\ldots(1)$$

$$f(\pi-)=\lim_{h\to 0} f(x-h)=\lim_{h\to 0} f(\pi-h)=-1$$

$$f(\pi+)=\lim_{h\to 0} f(x+h)=\lim_{h\to 0} 50=50$$

$$(1)\Rightarrow\ f(\pi)=\frac{-1+50}{2}=\frac{49}{2}$$

42. To which the value, the half range sine series corresponding to $f(x)=x^2$ **expressed in the interval** $(0,2)$ **converges at** $x=2$?

Solution:

Given $f(x)=x^2$

$x=2$ is a point of discontinuity and also it is an end point.

Since $f(x)=\begin{cases}x^2\ , & 0<x<2 \\ -x^2\ , & -2<x<0\end{cases}$

The half range sine series corresponding to $f(x)=x^2$ expressed in the interval $(0,2)$

converges at $x=2$ is $\frac{f(0)+f(2)}{2}$

$$\text{(I.e.,)}\quad \underset{at\,x=2}{f(x)}=\frac{f(0)+f(2)}{2}$$

$$=\frac{0+4}{2}$$

$$=2$$

43. If the Fourier series for the function $f(x)=\begin{cases} 0 & if\ 0<x<\pi \\ \sin x & if\ \pi<x<2\pi \end{cases}$

$$f(x)=-\frac{1}{\pi}+\frac{2}{\pi}\left[\frac{\cos 2x}{1.3}+\frac{\cos 4x}{3.5}+\frac{\cos 6x}{5.7}+......\right]+\frac{1}{2}\sin x$$

Deduce that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-......=\frac{\pi-2}{4}$ **[A.U MAY 2004]**

Solution:

Put $x=\frac{\pi}{2}$ is a point continuity.

$$0=-\frac{1}{\pi}+\frac{2}{\pi}\left[-\frac{1}{1.3}+\frac{1}{3.5}-\frac{1}{5.7}+......\right]+\frac{1}{2}$$

$$=-\frac{1}{\pi}-\frac{2}{\pi}\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-......\right]+\frac{1}{2}$$

$$\frac{1}{\pi}-\frac{1}{2}=-\frac{2}{\pi}\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-......\right]$$

$$\frac{2-\pi}{2\pi}=-\frac{2}{\pi}\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-......\right]$$

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-.....=\left[\frac{-(2-\pi)}{2\pi}\right]\times\frac{\pi}{2}$$

$$=\frac{\pi-2}{4}.$$

44. If $f(x)=\sqrt{1-\cos x}\ \ in\ \ 0\le x\le 2\pi$ **has a Fourier series expansion,**

$\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos nx$**, find the value of** $\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}$**.**

Solution: **Given:** $f(x)=\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos nx$.....................(1)

Put $x=0$ (is a point of continuity)

$$\left.\begin{matrix} f(x) \\ at\ x=0 \end{matrix}\right\}=\sqrt{1-\cos x}=\sqrt{1-1}=0$$

$\therefore$ (1) implies $0=\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}$

$$-\frac{2\sqrt{2}}{\pi}=-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}$$

$$\frac{1}{2}=\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}$$

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}=\frac{1}{2}$$

45. State the Dirichlet's conditions for the existence of the Fourier expansion of $f(x)$, in the interval $(c, c+2l)$. **[A.U.NOV 2011, N/D 2016, N/D 2017]**

Solution: Any function $f(x)$ can be developed as a Fourier series in $c \le x \le c+2l$

$\frac{a_o}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ where a_o , a_n , b_n are constants provided it satisfies

the following Dirichlet's conditions

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(c, c+2l)$.

(ii) $f(x)$ is periodic in $(c, c+2l)$.

(iii) $f(x)$ and $f^1(x)$ are piecewise continuous in $(c, c+2l)$.

(iv) $f(x)$ has no or finite number of maxima and minima $(c, c+2l)$.

46. If $(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ in $0 < x < 2\pi$ then deduct that $\sum_{n=1}^{\infty}\frac{1}{n^2}$

[AU NOV/DEC 2014]

Solution: **Let $f(x) = (\pi - x)^2$ and $x = 0$ is point of discontinuity**

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\because f(x) = \frac{f(0) + f(2\pi)}{2}$$

$$= \frac{\pi^2 + \pi^2}{2} \quad = \pi^2$$

47. If the fourier series of the function $f(x) = x$, $-\pi < x < x$ with period 2π is given by $f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right)$ then find the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

[AU A/M 2015]

Solution: let $x = \frac{\pi}{2}$ is point of continuity

$$\frac{\pi}{2} = 2\left(\sin \pi/2 - \frac{\sin 2(\pi/2)}{2} + \frac{\sin 3(\pi/2)}{3} - \frac{\sin 4(\pi/2)}{4} + \ldots\right)$$

$$\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = \frac{\pi}{4}$$

48. The instantaneous current "i" at time t of an alternating current wave is given by $i = I_1 \sin(\omega t + \alpha_1) + I_3 \sin(\omega t + \alpha_3) + I_5 \sin(\omega t + \alpha_5) + \ldots$ find the effective value of the current "i" **[A.U A/M 2015]**

Solution: $$i = \sum_{n=1}^{\infty} I_n \sin(\omega t + \alpha_n)$$

R.M.S value of $$i^2 = \frac{1}{\pi}\int_0^{\pi} [f(x)]^2\, dx$$

$$i = \sqrt{\frac{1}{\pi}\int_0^{\pi} [f(x)]^2\, dx}$$

(or)

$$i_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} [V(t)]^2\, dt}$$

49. Find the root mean square value of $f(x) = x(l-x)$ in $0 \le x \le l.$ **[A.U. N/D 2015]**

Solution: Given $f(x) = x(l-x)$

$$= \sqrt{\frac{\int_a^b [f(x)]^2\, dx}{b-a}}$$

$$= \sqrt{\frac{\int_0^l [x(l-x)]^2\, dx}{l-0}}$$

$$= \sqrt{\frac{\left(\frac{x^3}{3} l^2 - 2l\frac{x^4}{4} + \frac{x^5}{5}\right)}{l}}$$

$$= \sqrt{\frac{\left(\frac{l^5}{3} - 2\frac{l^5}{4} + \frac{l^5}{5}\right)}{l}}$$

$$= \frac{l^2}{\sqrt{30}}$$

50. Find the value of the Fourier series of $f(x)=\begin{cases}0 & in\ (-c,0)\\ 1 & in\ (0,c)\end{cases}$ **at the point of discontinuity x=0.** **[A.U. M/J 2016]**

Solution:

Given $f(x)=\begin{cases}0 & in\ (-c,0)\\ 1 & in\ (0,c)\end{cases}$ at x=0 is a discontinuity point,

$$f(x)=\frac{f(-c)+f(+c)}{2}$$

$$f(x)=\frac{0+1}{2}$$

$$=\frac{1}{2}$$

51. Find the value of b_n **in the Fourier series expansion of** $f(x)=\begin{cases}x+\pi & in\ (-\pi,0)\\ -x+\pi & in\ (0,\pi)\end{cases}$

[A.U. M/J 2016]

Solution: Given $f(x)=\begin{cases}x+\pi & in\ (-\pi,0)\\ -x+\pi & in\ (0,\pi)\end{cases}$

$$\varphi_1(x)=x+\pi$$

$$\varphi_2(x)=-x+\pi$$

$$\varphi_1(-x)=-x+\pi=\varphi_2(x)$$

$$\therefore f(x)\ is\ an\ even\ function$$

$$\therefore b_n=0.$$

52. If the Fourier series of the function $f(\text{x})=\text{x}+\text{x}^2$**, in the interval** $(-\pi,\pi)$ is $\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right]$ then find the value of the infinite series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.........$

Solution: **[A.U. A/M 2017]**

Given $f(\text{x})=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right]$

let $x=\pi$ is point of discontinuity

$$\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}=\frac{\pi+\pi^2-\pi+\pi^2}{2}=\pi^2$$

$$\sum_{n=1}^{\infty}\frac{4}{n^2}=\pi^2-\frac{\pi^2}{3}=\frac{2\pi^2}{3}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.........=\frac{\pi^2}{6}$$

PART-B

PROBLEMS IN THE INTERVAL $(0,2\pi)$

FORMULA : $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx\,,$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\ dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\ dx$$

1. Find the Fourier series expansion of f(x) = $\begin{cases}x & for\ 0<x\le\pi\\ 2\pi-x & for\ \pi\le x<2\pi\end{cases}$ **Also, deduce that**

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.........=\frac{\pi^2}{8}$$

Solution:

Formula: Fourier series $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\ \cos nx+\sum_{n=1}^{\infty}b_n\ \sin nx$

To find a_0:

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}x\,dx+\int_{\pi}^{2\pi}(2\pi-x)\,dx\right]=\frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi}+\left(2\pi x-\frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}+4\pi^2-\frac{4\pi^2}{2}-\left(2\pi^2-\frac{\pi^2}{2}\right)\right]=\frac{1}{\pi}\left[\frac{2\pi^2}{2}\right]$$

$$a_0=\pi$$

To find a_n:

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\ dx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}x\cos nx\,dx+\int_{\pi}^{2\pi}(2\pi-x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right)-\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}+\left[(2\pi-x)\left(\frac{\sin nx}{n}\right)-(-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{\pi}\left[\left(0+\frac{(-1)^n}{n^2}\right)-\left(0+\frac{1}{n^2}\right)+\left(0-\frac{1}{n^2}\right)-\left(0-\frac{(-1)^n}{n^2}\right)\right]=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

$$a_n = \begin{cases} \dfrac{-4}{\pi n^2} & if\ \ n\ is\ odd \\ 0 & if\ \ n\ is\ even \end{cases}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\ dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\ dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - \left(\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$b_n = 0$$

Hence required Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{\pi n^2}\cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x$$

2nd part:

Put x=0 is a point of discontinuity

$$f(x) = \frac{f(0) + f(2\pi)}{2} = \frac{0+0}{2} = 0$$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi}{2}$$

Result: $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots\ldots = \frac{\pi^2}{8}$

Hence Proved.

2. Find the Fourier series for f(x) $= \begin{cases} 1 \ \ for \ \ 0 < x < \pi \\ 2 \ \ for \ \ \pi < x < 2\pi \end{cases}$ **[AU Nov 2013]**

Solution: **Formula:** $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)....(1)$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$
$$= \frac{1}{\pi}\int_0^{\pi} f(x)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)dx$$
$$= \frac{1}{\pi}\int_0^{\pi} 1dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 2dx$$
$$= \frac{1}{\pi}[x]_0^{\pi} + \frac{1}{\pi}[2x]_{\pi}^{2\pi}$$
$$a_0 = 3$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 2\cos nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$$
$$a_n = 0$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 2\sin nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}$$
$$b_n = \frac{(-1)^n - 1}{n\pi}$$
$$b_n = \begin{cases} \frac{-2}{n\pi}, \ n \ is \ odd \\ 0 \quad , \ n \ is \ even \end{cases}$$
$$(1) \Rightarrow \ f(x) = \frac{3}{2} - \frac{2}{n\pi}\sum_{n=odd}^{\infty} \cos nx$$

3. Find the Fourier Series of period 2π for the function $f(x)=x\cos x$ in $0<x<2\pi$

[A.U. A/M 2017]

Solution: Given $f(x)=x\cos x$

Formula: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx$

To find a_0:

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x\cos x\,dx$$

$$=\frac{1}{\pi}\left[x(\sin x)-(1)(-\cos x)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}(1-1)$$

$$a_0=0$$

To find a_n:

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x\cos x\cos nx\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n+1)x+\cos(n-1)x\right]dx$$

$$\textit{Consider}\quad \int_0^{2\pi} x\cos nx\,dx=\left[\left(\frac{\sin nx}{n}\right)+\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$=\frac{1}{n^2}\left[\cos 2n\pi-\cos 0\right]=0$$

$$a_n=\frac{1}{2\pi}\left[0+0\right]=0,\ n\neq 1 \text{------------------}(1)$$

Now, $a_n=\frac{1}{\pi}\int_0^{\pi} x\cos x\cos nx\,dx$

$$\textit{But}\quad a_1=\frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x\left(\frac{1+\cos 2x}{2}\right)dx$$

$$=\frac{1}{2\pi}\left[\int_0^{2\pi} x\,dx+\int_0^{2\pi} x\cos 2x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} + 0 \qquad using \quad (1)$$

$$= \left[\frac{4\pi^2}{4\pi}\right] = \pi$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos x\sin nx dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x + \sin(n-1)x\right] dx$$

$$Consider \int_0^{2\pi} x \ \text{sinn}\, x \ dx = \left[\left(\frac{-\text{xcos}\, nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{n}\left[-2\pi\cos 2n\pi\right] = \frac{-2\pi}{n}$$

Now, $$b_n = \frac{1}{2\pi}\left[\frac{-2\pi}{n+1} - \frac{2\pi}{n-1}\right]$$

$$= -\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= -\left[\frac{2n}{n^2-1}\right] \quad for \quad n \neq 1$$

To find b_1: $$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x\sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x \ dx$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{2}\right] = -\frac{1}{2}$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + + \sum_{n=2}^{\infty} a_n \ \cos nx + \sum_{n=2}^{\infty} b_n\sin nx$$

Result: $$f(x) = \pi\cos x - \frac{\sin x}{2} - 2\sum_{n=2}^{\infty}\frac{n\sin nx}{n^2-1}$$

PROBLEMS IN THE INTERVAL $(-\pi,\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos n\;x\;dx$$

4. Find the Fourier series of the periodic function defined by $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Deduce that $\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{1^2}+\ldots+\infty = \frac{\pi^2}{8}$ **[A.U NOV 2009]**

Solution: Given $f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$

for $-\pi < x < 0,\ f(-x) = -\pi \neq f(x)$

for $0 < x < \pi,\ f(-x) \neq -f(x)$

Therefore f(x) is neither even nor odd.

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi\left(0-(-\pi)\right) + \left(\frac{\pi^2}{2}-0\right)\right]$$

$$= \frac{1}{\pi}\left[-\pi(\pi) + \left(\frac{\pi^2}{2}\right)\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$$

$$a_0 = \frac{-\pi}{2}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\cos nx dx + \frac{1}{\pi}\int_0^{\pi} x\cos nx dx$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x\left(\frac{\sin nx}{n}\right) - 1.\left(-\frac{\cos nx}{n^2}\right)\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{1}{\pi n^2}[(-1)^n - 1]$$

$$a_n = \begin{cases} \dfrac{-2}{\pi n^2} & if \;\; n \; is \; odd \\ 0 & if \;\; n \; is \, even \end{cases}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-\pi)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left(x\left(\frac{-\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right)_0^{\pi}\right]$$

$$= \frac{\pi}{\pi n}\left[1 - (-1)^n - (-1)^n\right]$$

$$b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$$\therefore\; f(x) = \frac{\left(-\frac{\pi}{2}\right)}{2} + \sum_{n=0dd}^{\infty} \frac{-2}{\pi n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin nx \text{ --------- }(1)$$

Deduction: Put x=0 in f(x),

We get at x=0 is a point of discontinuity in the middle.

$$\therefore\; f(x) \;\; at \;\;\; x = 0\} = \frac{f(0-) + f(0+)}{2}$$

$$f(x) = \frac{(-\pi + 0)}{2}$$

$$f(x) = -\frac{\pi}{2}$$

$$(1) \Rightarrow -\frac{\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\sum_{n=0dd}^{\infty}\frac{1}{n^2}.1 + \sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right].0$$

$$-\frac{\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\sum_{n=0dd}^{\infty}\frac{1}{n^2}$$

$$-\frac{\pi}{2} + \frac{\pi}{4} = -\frac{2}{\pi}\sum_{n=0dd}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

$$ie., \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... + \infty = \frac{\pi^2}{8}$$

Hence proved.

5. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $(-\pi, \pi)$.

[A.U. Nov 2000]

Solution: Given $f(x) = x \sin x$ is an even function

$$\therefore b_n = 0$$

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

Let the required Fourier series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}(\pi)$$

$$= 2$$

$$a_0 = 2$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx \sin x \, dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) - \left(\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1}\right) - (-0-0)\right] - [0-0+0]$$

$$= (-1)^n \left[\frac{n-1-n-1}{n^2-1}\right]$$

$$= (-1)^n \left[\frac{-2}{n^2-1}\right] \ , \ n \neq 1$$

$$a_n = \left[\frac{-2}{n^2-1}\right](-1)^n \ , \ n \neq 1$$

$$a_n = (-1)^{n+1}\left[\frac{2}{n^2-1}\right], \ n \neq 1$$

To find a_1: $$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x \sin 2x \ dx$$

$$= \frac{1}{\pi}\left[x\frac{(-\cos 2x)}{2} - (1)\frac{(-\sin 2x)}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-x\frac{(\cos 2x)}{2} + \frac{(\sin 2x)}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi}{2}+0\right) - (0+0)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\right] = -\frac{1}{2}$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \ \cos nx$$

$$= \frac{2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} (-1)^{(n+1)}\left[\frac{2}{(n^2-1)}\right]\cos nx$$

$$= 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} (-1)^{(n+1)}\left[\frac{1}{(n^2-1)}\right]\cos nx$$

Result: $$f(x) = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \dots\right]$$

6. Find the Fourier series of $f(x) = x \ in \ -\pi < x < \pi.$ [A.U. M/J 2016]

Solution: Let $f(x) = x \ , \ f(-x) = -x = -f(x)$

Therefore $f(x)$ *is an odd function in* $(-\pi, \pi)$

$$Hence \quad a_0 = 0 \ \ and \ \ a_n = 0 \quad for\,all \ \ n > 0,$$

Formula: $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

Let the required Fourier series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $b_n = \dfrac{2}{\pi} \sum_{n=1}^{\infty} f(x) \sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

By applying Bernoulli's formula

$$u = x \quad ; \quad v = \sin nx$$

$$u' = 1 \quad ; \quad v_1 = \frac{-\cos nx}{n}$$

$$u'' = 0 \quad ; \quad v_2 = \frac{-\sin nx}{n^2}$$

$$\int uv\, dx = uv - u' v_1 + u'' v_2 - \ldots$$

$$= \frac{2}{\pi}\left[-x\left(\frac{\cos nx}{n} + \left(\frac{\sin nx}{n^2}\right)\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi(-1)^n}{n} + 0\right) - (0+0)\right]$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

Result: $f(x) = 2\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin nx$

7. Find the Fourier series for $f(x) = |\cos x|$ **in the interval** $(-\pi, \pi)$**.** **[A.U M/J 2016]**

Solution:

Given $f(x) = |\cos x|$

f(x) is an even function , $b_n = 0$

Formula: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx -----(1)$$

To find a_0:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi}|\cos x|dx$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\cos x\, dx+\frac{2}{\pi}\int_{\pi/2}^{\pi}-\cos x\, dx$$

$$=\frac{2}{\pi}\left\{[\sin x]_0^{\pi/2}+[-\sin x]_{\pi/2}^{\pi}\right\}$$

$$=\frac{2}{\pi}[1+1]$$

$$a_0=\frac{4}{\pi}$$

To find a_n:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$=\frac{2}{\pi}\int_0^{\pi}|\cos x|\cos nx\, dx$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\cos x\ \cos nx\, dx+\frac{2}{\pi}\int_{\pi/2}^{\pi}-\cos x\ \cos nx\, dx$$

$$=\frac{1}{\pi}\left\{\int_0^{\pi/2}[\cos(n+1)x+\cos(n-1)x]dx-\int_{\pi/2}^{\pi}[\cos(n+1)x+\cos(n-1)x]dx\right\}$$

$$=\frac{1}{\pi}\left\{\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2}-\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}\right\}$$

$$=\frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$=\frac{2\cos\frac{n\pi}{2}}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{2\cos\frac{n\pi}{2}}{\pi}\left[\frac{-2}{n^2-1}\right] \qquad a_n=-\frac{4\cos\frac{n\pi}{2}}{\pi(n^2-1)} \quad when\ n\neq 1$$

When n=1,

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x| \cos x \, dx$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \frac{1+\cos 2x}{2} dx - \int_{\pi/2}^{\pi} \frac{1+\cos 2x}{2} dx\right]$$

$$= \frac{1}{\pi}\left\{\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2} - \left[x+\frac{\sin 2x}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$a_1 = 0$$

Result: $$f(x) = \frac{2}{\pi} - \sum_{n=2}^{\infty} \frac{4\cos\frac{n\pi}{2}}{\pi(n^2-1)}\cos nx$$

8. Find the Fourier series expansion of $f(x) = x + x^2 \;\; in \;\; (-\pi,\pi)$ and hence deduce the value of $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ **[A.U. N/D 17]**

Solution: Given $f(x) = x + x^2$

$$f(-x) = -x + x^2$$

$$f(-x) \neq f(x)$$

Therefore, $f(x)$ is neither even nor odd in $(-\pi,\pi)$

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 dx \qquad \text{since } x \text{ is odd and } x^2 \text{ is even}$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2\pi^2}{3}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx dx \qquad [\because x\cos nx \; is \; odd \;\; x^2\cos nx \;\; is \;\; even]$$

$$= \frac{2}{\pi}\left\{x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}_0^{\pi}$$

$$a_n = \frac{4}{n^2}(-1)^n$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx dx \qquad [\because x\sin nx \; is \; even \;\; x^2\sin nx \;\; is \;\; odd]$$

$$= \frac{2}{\pi}\left\{x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi}$$

$$b_n = \frac{-2}{n}(-1)^n$$

$$(1) \Rightarrow f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin nx$$

Result: $$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$$

Deduction:

$x = \pi$

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{f(\pi)+f(-\pi)}{2}$$

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi+\pi^2+\pi^2-\pi}{2} = \frac{2\pi^2}{2} = \pi^2$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{3\pi^2-\pi^2}{3} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6}$$

9. Find the Fourier series for $f(x)=x^2 \quad in \quad -\pi<x<\pi$. **Hence show that**

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+...=\frac{\pi^4}{90}$$

[A.U. 2013, Nov 2014, N/D 2016, A/M 18]

Solution:

Given : $f(x)=x^2$

$$f(-x)=(-x)^2=x^2=f(x)$$

Therefore f(x) is an even function in (-π ,π). $b_n = 0$

Formula: $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$$

∴ The Fourier series of is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$

To find a_0:

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}x^2dx$$

$$=\frac{2}{\pi}\left[\frac{x^2}{3}\right]_0^{\pi} \qquad a_0=\frac{2\pi^2}{3}$$

To find a_n:

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx$$

$$=\frac{2}{\pi}\left[\left(x^2\right)\left(\frac{\sin nx}{n}\right)-\left(2x\right)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$a_n=\frac{4}{n^2}(-1)^n$$

$$f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

Parseval identity is

$$\frac{2}{\pi}\int_{-\pi}^{\pi}\left[f(x)\right]^2dx=\frac{a_0^{\,2}}{2}+\sum_{n=1}^{\infty}a_n^2$$

$$\therefore \int_{-\pi}^{\pi} \left[x^2\right]^2 dx = \frac{\pi}{2}\left[\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

$$\left(\frac{x^5}{5}\right)_{-\pi}^{\pi} = \frac{\pi}{2}\left[\frac{1}{2}\left(\frac{4\pi^4}{9}\right) + \sum_{n=1}^{\infty}\frac{16}{n^4}(-1)^{2n}\right]$$

$$\frac{\pi^5}{5} = \frac{\pi}{2}\left[\left(\frac{2\pi^4}{9}\right) + 16\sum_{n=1}^{\infty}\frac{1}{n^4}\right]$$

$$\frac{2\pi^4}{5} - \frac{2\pi^4}{9} = \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{8\pi^4}{45} = \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$$

Hence proved.

10 . **If** $f(x) = \begin{cases} 1-x, -\pi \le x \le 0 \\ 1+x, 0 \le x \le \pi \end{cases}$ **find the Fourier series of f(x) and hence deduce that** $\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{8}$ **[A.U. MAY 2011, NOV 2013]**

Solution:

Given that

f(x) = $\begin{cases} 1-x & for\ -\pi \le x \le 0 \\ 1+x & for\ 0 \le x \le \pi \end{cases}$, which is an even function $b_n = 0$

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

So, the required Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx ----------(1)$$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(1+x)\,dx$$

$$= \frac{2}{\pi}\left[x + \frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = 2 + \pi$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (1+x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(1+x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} -\dfrac{4}{\pi n^2} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

$$f(x) = \frac{2+\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}\cos nx \text{ ---------- } (2)$$

2 nd part:

Put x=0, f(x) =1

$$1 = \frac{2+\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}$$

$$1 - \frac{2+\pi}{2} = \frac{-4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\left(\frac{2-2-\pi}{2}\right)\left(-\frac{\pi}{4}\right) = \sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{8}$$

Hence proved.

11. Find the Fourier series of $f(x) = |\sin x|\ \ in\ -\pi < x < \pi$ of periodicity 2π **.[AU A/M 2015]**

Solution: Given $f(x) = |\sin x| = \begin{cases} \sin x & 0 \le x \le \pi \\ -\sin x & -\pi \le x \le 0 \end{cases}$

$$f(-x) = |\sin(-x)| = |\sin x| = f(x)$$

Therefore $f(x)$ is even function .hence $b_n = 0$

Formula: $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

Let the required fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ ----------------**(1)**

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\ dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$a_0 = \frac{4}{\pi}$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\ dx$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} \left[\sin (n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right][1 + (-1)^n]$$

$$= \frac{-2}{\pi(n^2-1)}[1 + (-1)^n]\ , \qquad n \neq 1$$

$$a_n = \begin{cases} \dfrac{-4}{\pi(n^2-1)} & if\ \ n\ is\ even \\ 0 & f\ \ n\ is\ odd \end{cases}$$

To find a_1**:**

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x\ \cos x\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\sin 2x}{2}\right)dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{2\pi}\left[-\cos 2x\right]_0^{\pi}$$

$$a_1 = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

Result:
$$f(x) = \frac{2}{\pi} + \sum_{n=even}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$$
$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=even}^{\infty} \frac{\cos nx}{(n^2-1)}$$

PROBLEMS IN THE INTERVAL $(-l, l)$

12. Find the Fourier series expansion the following periodic function of period 4

$f(x) = \begin{cases} 2+x & -2 \le x \le 0 \\ 2-x & 0 < x \le 2 \end{cases}$ **. Hence deduce that** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. **[A.U. N/D 2015]**

Solution: Let $\varphi_1(x) = 2 + x$; $\varphi_2(x) = 2 - x$

$$f(x) = \begin{cases} \varphi_1(x) \ ; & -2 \le x \le 0 \\ \varphi_2(x) \ ; & 0 < x \le 2 \end{cases}$$
$$\varphi_1(-x) = 2 - x = \varphi_2(x)$$
$$\therefore f(x) \text{ is an even function}$$
$$\therefore b_n = 0.$$

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Let the required Fourier series be
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

To find a_0:
$$a_0 = \frac{2}{l} \int_0^2 f(x)\, dx$$
$$= \frac{2}{2} \int_0^2 (2-x)\, dx$$
$$= \left[2x - \frac{x^2}{2} \right]_0^2$$
$$= 4 - \frac{4}{2}$$
$$= 4 - 2$$
$$a_0 = 2$$

To find a_n:
$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$
$$= \frac{2}{2} \int_0^2 (2-x) \cos \frac{n\pi x}{2}\, dx$$

$$= \int_0^2 (2-x)\cos\frac{n\pi x}{2}\,dx$$

By applying Bernoulli's equation

Here $\quad u = 2-x \quad ; \quad \int dv = \cos\frac{n\pi x}{2}$

$$u' = -1 \quad ; \quad v = \sin\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)$$

$$v_1 = -\cos\frac{n\pi x}{2}\left(\frac{2}{n\pi}\right)^2$$

$$= \left[\frac{2}{n\pi}(2-x)\left(\sin\frac{n\pi x}{2}\right) - \left(\cos\frac{n\pi x}{2}\right)\left(\frac{4}{n^2\pi^2}\right)\right]_0^2$$

$$= \left[\frac{-4}{n^2\pi^2}\left(\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \left[\frac{-4}{n^2\pi^2}(\cos\pi - \cos 0)\right]$$

$$= \left[\frac{-4}{n^2\pi^2}\left((-1)^n - 1\right)\right]$$

$$a_n = \begin{cases} \dfrac{8}{n^2\pi^2} & ; \ if \ n \ is \ odd \\ 0 & ; \ if \ n \ is \ even \end{cases}$$

$$a_n = \frac{8}{n^2\pi^2}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=odd}^{\infty} a_n \cos nx$$

$$(2-x) = \frac{2}{2} + \frac{8}{\pi^2}\sum_{n=odd}^{\infty} \frac{\cos nx}{n^2}$$

Put x=0

$$2 = 1 + \frac{8}{\pi^2}\sum_{n=odd}^{\infty} \frac{\cos 0}{n^2}$$

$$1 = \frac{8}{\pi^2}\sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=odd}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Result: $\quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Hence proved.

PROBLEMS IN THE INTERVAL $(0,\pi)$

Formula $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\ dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$\frac{2}{\pi}\int_0^{\pi}\left[f(x)\right]^2 dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty} a_n^2$$

13. Find the half-range cosine series of $f(x)=(\pi-x)^2$ in (0, π). Hence find the sum of the Series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.............\infty$ **[A.U NOV 2009 , N/D 2015]**

Solution: Given $f(x)=(\pi-x)^2$

Formula: $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx----------(1)$

To find a_0:

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\ dx$$

$$=\frac{2}{\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{\pi}$$

$$a_0=\frac{2\pi^2}{3}$$

To find a_n:

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\cos nx\, dx$$

$$=\frac{2}{\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right)-(-2)(\pi-x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

2nd part

By Parseval's identity

$$\frac{2}{\pi}\int_0^{\pi}\left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{1}{\pi}\int_0^{\pi}\left[\pi - x\right]^4 dx = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{1}{\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^{\pi} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{5} - \frac{\pi^4}{9} = 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Result: $$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \frac{\pi^4}{90}$$

14. Find the half-range cosine series of the function $f(x) = x(\pi - x)$ in the interval $0 < x < \pi$. Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$. [A.U. A/M 2018]

Solution: Given $f(x) = x(\pi - x)$ in $0 < x < \pi$.

$$f(x) = \pi x - x^2$$

Formula: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(x\pi - x^2\right) dx$$

$$= \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0-0)\right]$$

$$= \frac{2}{\pi}\pi^3\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{2}{\pi}\pi^3\left[\frac{3-2}{6}\right]$$

$$a_0 = \frac{\pi^2}{3}$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\pi x - x^2\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\pi x - x^2\right)\left[\frac{\sin nx}{n}\right] - (\pi - 2x)\left[\frac{-\cos nx}{n^2}\right] + (-2)\left[\frac{-\sin nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi x - x^2\right)\left[\frac{\sin nx}{n}\right] + (\pi - 2x)\left[\frac{\cos nx}{n^2}\right] + 2\left[\frac{\sin nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(0 - \frac{\pi\cos n\pi}{n^2} + 0\right) - \left(0 + \frac{\pi}{n^2} + 0\right)\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n^2} - \frac{\pi}{n^2}\right]$$

$$= -\left(\frac{2}{\pi}\right)\left(\frac{\pi}{n^2}\right)\left[\cos n\pi + 1\right]$$

$$= -\left(\frac{2}{n^2}\right)\left[(-1)^n + 1\right]$$

$$\therefore a_n = \begin{cases} 0 \text{ if } n \text{ is odd} \\ \dfrac{-4}{n^2} \text{ if } n \text{ is even} \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi^2}{6} - 4\sum_{n=2,4,\ldots}^{\infty}\frac{1}{n^2}\cos nx$$

Using parseval's identity

$$\frac{2}{\pi}\int_0^{\pi}\left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{a_0^2}{4}=\frac{\pi^4}{36}$$

$$\frac{\pi^4}{36}+\frac{1}{2}\sum_{n=even}^{\infty}\frac{16}{(2n)^4}=\frac{1}{\pi}\int_0^{\pi}\left[\pi x-x^2\right]^2\,dx$$

$$\frac{\pi^4}{36}+\frac{1}{2}\sum_{n=even}^{\infty}\frac{1}{n^4}=\frac{1}{\pi}\int_0^{\pi}\left(\pi^2x^2-2\pi x^3+x^4\right)\,dx$$

$$=\frac{1}{\pi}\left[\left(\pi^2\frac{x^3}{3}-2\pi\frac{x^4}{4}+\frac{x^5}{5}\right)\right]_0^{\pi}\,dx$$

$$=\frac{1}{\pi}\left[\left(\frac{\pi^5}{3}-\frac{\pi^5}{2}+\frac{\pi^5}{5}\right)\right]$$

$$=\frac{\pi^4}{30}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{30}-\frac{\pi^4}{36}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{15}-\frac{\pi^4}{18}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...=\frac{\pi^4}{90}$$

Hence proved.

15. Obtaion the half-range cosine series for $f(x)=x \quad in \quad (0,\pi)$ **and hence Prove that** $\frac{1}{1^2}+\frac{1}{3^2}+...$ **[A.U N/D 2017]**

Solution: The half range cosine series is given by

Formula: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$

To find a$_0$:

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\,dx$$

$$=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\pi$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\ dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin\ nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x\frac{\sin\ nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left[x\sin\ nx + \frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left[\left(0 + \frac{(-1)^n}{n}\right) - \left(0 + \frac{1}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n}\right] \qquad = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0 & if\quad n\ \ is\ even \\ \dfrac{-4}{n^2\pi} & if\ \ n\ \ is\ \ odd \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

Result: $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\sum\limits_{n=odd}^{\infty} \dfrac{1}{(2n-1)^2}\cos(2n-1)x$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

$$x = \pi,\ \ \pi = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}\cos n\pi$$

$$\pi = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4(-1)^n}{n^2\pi}$$

$$\pi = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{4}{n^2\pi}$$

$$\pi - \frac{\pi}{2} = \sum_{n=odd}^{\infty} \frac{4}{n^2\pi}$$

$$\frac{\pi^2}{8} = \sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$$

16. Expand f(x) = $\begin{cases} x & for\ 0 < x \le 1 \\ 2-x & for\ 1 \le x < 2 \end{cases}$ **as a series of cosines in the interval (0,2).**

Solution: **[A.U. N/D 2016, A/M 2017]**

Given f(x) = $\begin{cases} x & for\ 0 < x \le 1 \\ 2-x & for\ 1 \le x < 2 \end{cases}$

Formula: Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$

Here l=2

To find a_0:

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx$$

$$= \left[\int_0^1 x\,dx + \int_1^2 (2-x)\,dx\right] = \left[\left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2\right]$$

$$= \left[\frac{1}{2} + 4 - \frac{4}{2} - \left(2 - \frac{1}{2}\right)\right] = \left[\frac{2}{2}\right]$$

$$a_0 = 1$$

To find a_n:

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\int_0^1 x\cos\frac{n\pi x}{2}\,dx + \int^2 (2-x)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \left\{\left[\frac{2}{n\pi}x\left(\sin\frac{n\pi x}{2}\right) - \left(-\left[\frac{2}{n\pi}\right]^2\cos\frac{n\pi x}{2}\right)\right]_0^1 + \left[\frac{2}{n\pi}(2-x)\left(\sin\frac{n\pi x}{2}\right) - (-1)\left(-\cos\frac{n\pi x}{2}\right)\right]_1^2\right\}$$

$$= \frac{4}{\pi^2 n^2}\left[2\cos\frac{n\pi}{2} - (-1)^n - 1\right]$$

$$a_n = 0 \quad \text{if n is odd}$$

$$a_2 = \frac{4}{\pi^2 n^2}\left[2\cos\pi - 1 - 1\right] = \frac{-4}{\pi^2}$$

$$a_4 = \frac{4}{\pi^2 n^2}\left[2\cos 2\pi - 1 - 1\right] = 0$$

$$a_6 = \frac{4}{\pi^2 n^2}\left[2\cos 3\pi - 1 - 1\right] = \frac{-16}{6^2\pi^2}$$

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\pi x + 0 + \frac{1}{9}\cos\frac{6\pi x}{2}\right]$$

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\pi x + \frac{1}{9}\cos 3\pi x\right]$$

Result:

17. Find the half range cosine series of f(x) = x in 0 < x< π. **[A.U DEC 2010]**

Solution:

Half range cosine series is

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \; dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx \; dx = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \dfrac{-4}{\pi n^2} \; if \;\; n \; is \; odd \\ 0 \;\; if \;\; n \; is \; even \end{cases}$$

Required half range cosine series is

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{\pi\, n^2}\cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\cos nx$$

Result:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x$$

18. Find the half range sine series of $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$ **. Hence deduce the sum of the series** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ **.** **[AU A/M 2015]**

Solution: Let the half range sine series be

$$f(x) = \sum_{n=1}^{b} b_n \sin nx \qquad ------(1)$$

To find b_n:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\ nx\ dx$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x}{2}\left(\frac{\cos\frac{nx}{2}}{n}\right) + \left(\frac{\sin\frac{nx}{2}}{n^2}\right) - (-0+0)\right] - \frac{2}{\pi}\left[(0+0) - \left(\frac{\pi}{2}\right)\left(\frac{\cos\frac{nx}{2}}{n}\right) + \left(\frac{\sin\frac{nx}{2}}{n^2}\right)\right]$$

$$= \frac{2}{\pi}2\left(\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}\right)$$

$$b_n = \begin{cases} \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

Deduction

$$f(x) = \sum_{n=odd}^{\infty} \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)\sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi}\sin\left(\frac{(2n-1)\pi}{2}\right)\sin(2n-1)x$$

Result: $$f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$$

PROBLEMS IN THE INTERVAL $(0,l)$

19. Find the half range sine series $f(x) = lx - x^2\ \ in\ (0,l)$ **[AU Nov 2013]**

Solution:

Formula: $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\ldots.(1)$$

To find b_n:

$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(lx - x^2\right)\left(\frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l}\right) - \left(l - 2x\right)\left(\frac{-\sin\left(\frac{n\pi x}{l}\right)}{n^2\pi^2/l^2}\right) + \left(-2\right)\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-2l^3\left(-1\right)^n}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4l^2}{n^3\pi^3}\left[1 + \left(-1\right)^n\right]$$

$$b_n = \begin{cases} \dfrac{8l^2}{n^3\pi^3}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

Result:

$$(1) \Rightarrow f(x) = \frac{8l^2}{n^3\pi^3}\sum_{n=even}^{\infty} \sin\left(\frac{n\pi x}{l}\right)$$

20. Find the half-range cosine series for $f(x) = \left(x - 1\right)^2$ in $0 < x < 1.$

Solution: **[A.U. MAY 2013, Nov 2014]**

The required Fourier cosine series be

Formula: $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos n\pi x$.............(1)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= 2\int_0^1 \left(x - 1\right)^2 dx$$

$$= \left[\frac{\left(x - 1\right)^3}{3}\right]_0^1$$

$$= 2\left[0 - \frac{\left(-1\right)^3}{3}\right] \quad = \frac{2}{3}$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \; dx$$

$$= 2\int_0^1 (x-1)^2 \cos n\pi x dx$$

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$(1) \Rightarrow f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$$

Result:

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x$$

21. Obtain fourier cosine series of $f(x) = x,\ 0 < x < 4$ **hence deduct that** $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... + \infty$

Solution: Given $f(x) = x,\ 0 < x < 4$ **[AU NOV 2014]**

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

Here $l = 4$

To find a_0:

$$a_0 = \frac{2}{l}\int_0^l f(x) dx$$

$$= \frac{2}{4}\int_0^4 x\, dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2}\left[\frac{4^2}{2} - 0\right]$$

$$= \frac{1}{2}[8]$$

$$a_0 = 4$$

To find a_n:

$$a_n = \frac{2}{l}\int_0^l x \cos\left(\frac{n\pi x}{l}\right) dx$$

$$a_n = \frac{2}{4}\int_0^4 x \cos\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2}\left[x\left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi x}{4}\right) + \left(\frac{4}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= \frac{1}{2}\left[\left(0 + \frac{16}{n\pi}(-1)^n\right) - \left(0 + \frac{16}{n^2\pi^2}\right) \right]$$

$$= \frac{16}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$a_n = \begin{cases} 0 & if\ n\ is\ even \\ \frac{-16}{n^2\pi^2} & if\ n\ is\ odd \end{cases}$$

By Parseval's identity

$$\frac{2}{l}\int_0^l \left[f(x)\right]^2 dx = \frac{a_0^{\,2}}{2} + \sum_{n=1}^{\infty} a_n^{\,2}$$

$$\frac{1}{4}\int_0^4 x^2\, dx = \frac{16}{4} + \frac{1}{2}\sum_{n=odd} \frac{(16)^2}{n^4\pi^4}$$

$$\frac{1}{4}\left(\frac{x^3}{3}\right)_0^4 = 4 + \frac{128}{\pi^4}\sum_{n=odd}\frac{1}{n^4}$$

$$\frac{1}{4}\left(\frac{4^3}{3} - 0\right) - 4 = \frac{128}{\pi^4}\sum_{n=odd}\frac{1}{n^4}$$

$$\frac{16}{3} - 4 = \frac{128}{\pi^4}\sum_{n=odd}\frac{1}{n^4}$$

$$\frac{4}{3} = \frac{128}{\pi^4}\sum_{n=odd}\frac{1}{n^4}$$

Result: $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$

22. Obtain the Fourier series for the function f(x)= $\begin{cases} x & for\ 0 \le x \le 1 \\ 2-x & for\ 1 \le x \le 2 \end{cases}$

(or) Expand f(x)= $\begin{cases} x & for\ 0 \le x \le 1 \\ 2-x & for\ 1 \le x \le 2 \end{cases}$ **as a series of cosine in the interval (0,2)**
[A.U MAY 2011, 2013,N/D 2016, A/M 17]

Solution:

Given f(x)= $\begin{cases} x & for\ 0 \le x \le 1 \\ 2-x & for\ 1 \le x \le 2 \end{cases}$

Here $l=1$

Formula: $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

To Find a_0:

$$a_0=\frac{1}{l}\int_0^{2l}f(x)dx$$
$$=\frac{1}{1}\int_0^{2}f(x)dx$$
$$=\int_0^1 f(x)dx+\int_1^2 f(x)dx$$
$$=\int_0^1 xdx+\int_1^2(2-x)dx$$
$$=\left[\frac{x^2}{2}\right]_0^1+\left[2x-\frac{x^2}{2}\right]_1^2$$
$$=\left[\frac{1}{2}-0\right]+\left[(4-2)-\left(2-\frac{1}{2}\right)\right]$$
$$a_0=1$$

To find a_n:

$$a_n=\frac{1}{l}\int_0^2 f(x)\cos\frac{n\pi x}{l}dx$$
$$=\int_0^1 x\cos n\pi x dx+\int_1^2(2-x)\cos n\pi x dx$$
$$=\left[\left[(x)\frac{\sin n\pi x}{n\pi}+(1)\left(\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_0^1+\left[(2-x)\frac{\sin n\pi x}{n\pi}+(-1)\left(\frac{\cos n\pi x}{(n\pi)^2}\right)\right]_1^2\right]$$
$$=\frac{(-1)^n}{n^2\pi^2}-\frac{1}{n^2\pi^2}+\frac{(-1)^n}{n^2\pi^2}-\frac{1}{n^2\pi^2}$$
$$a_n=2\left[\frac{(-1)^n-1}{n^2\pi^2}\right]$$
$$a_n=\begin{cases}0 \quad , \quad if \quad n \ is \ even\\ \dfrac{-4}{n^2\pi^2}, \ if \ \ n \ is \ odd\end{cases}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$$

$$= \int_0^1 x\sin n\pi x dx + \int_1^2 (2-x)\sin n\pi x dx$$

$$= \left[\left[(x)\frac{-\cos n\pi x}{n\pi} + (1)\left(\frac{\sin n\pi x}{(n\pi)^2}\right)\right]_0^1 + \left[(2-x)\frac{-\cos n\pi x}{n\pi} + (-1)\left(\frac{\sin n\pi x}{(n\pi)^2}\right)\right]_1^2\right]$$

$$= \frac{-\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi}$$

$$b_n = 0$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} 0 \sin\frac{n\pi x}{l}$$

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

23. Obtain the Fourier series for the function $f(x)=\begin{cases} l-x & for\ 0 \le x \le l \\ 0 & for\ l \le x \le 2l \end{cases}$

[A.U N/D 17]

Solution:

Given $f(x)=\begin{cases} l-x & for\ 0 \le x \le l \\ 0 & for\ l \le x \le 2l \end{cases}$

Formula: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

To Find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$= \frac{1}{l}\left[\int_0^1 f(x)dx + \int_1^2 f(x)dx\right]$$

$$= \frac{1}{l}\left[\int_0^l (l-x)dx\right]$$

$$= \frac{1}{l}\left[\frac{(l-x)^2}{-2}\right]_0^l$$

$$= \frac{1}{-2l}\left[-l^2\right]$$

$$a_0 = \frac{l}{2}$$

To find a_n:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left(\int_0^{l}(l-x)\cos\frac{n\pi x}{l}dx\right)$$

$$= \frac{1}{l}\left[\left[(l-x)\frac{\sin n\pi x/l}{\frac{n\pi}{l}} - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l\right]$$

$$= \frac{-(-1)^n}{n^2\pi^2} + \frac{1}{n^2\pi^2}$$

$$a_n = \begin{cases} 0 \quad , \quad if \quad n \ is \ even \\ \dfrac{2l}{n^2\pi^2}, \ if \ \ n \ is \ odd \end{cases}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[\int_0^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{1}{l}\left[\left[(l-x)\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1)\left(\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l\right]$$

$$= \frac{1}{n\pi}$$

$$f(x) = \frac{l}{4} + \sum_{n=1,3,5,...}^{\infty}\frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\right]sin\frac{n\pi x}{l}$$

24. Obtain the sine series for the function $f(x) = \begin{cases} x \ \ for \ 0 \le x \le l/2 \\ l-x \ \ for \ l/2 \le x \le l \end{cases}$

Solution: **[A.U. May 2011, 2013]**

The sine series for the function $f(x)$ in $(0,l)$ is given by

Formula: $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

To find b_n:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi x}{l}\right)-\left(\frac{l^2}{n^2\pi^2}\right)\left(-\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+\frac{2}{l}\left[(l-x)\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi x}{l}\right)+(1)\left(\frac{l^2}{n^2\pi^2}\right)\left(-\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\left(\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)-(-0+0)\right]$$

$$+\frac{2}{l}\left[(-0-0)-\left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{l}\left[2\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] \qquad \Rightarrow b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{l}$$

$$= \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{l}$$

$$f(x)=0 \textit{ when } n \textit{ is even} \quad f(x) = \frac{4l}{\pi^2}\sum_{n=odd}^{\infty}\left[\frac{1}{n^2}\sin\frac{n\pi}{2}\right]\sin\frac{n\pi x}{l}$$

$$f(x) = \frac{4l}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\sin\frac{(2n+1)\pi}{2}\right]\sin\frac{(2n+1)\pi x}{l}$$

Result: $f(x) = \frac{4l}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(2n+1)^2}\right]\sin\frac{(2n+1)\pi x}{l}$

25. Find the Fourier series expansion of $f(x)=1-x^2$ in the interval (-1,1) [A.U DEC 2010]

Solution:

Given $f(x) = 1-x^2$ in (-1,1), Which is an even function hence $b_n=0$

The required Fourier series is

Formula: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

Here $2l = 2 \Rightarrow l = 1$

To find a_0:

$$a_0 = \frac{2}{1}\int_0^1 f(x)\, dx$$

$$= 2\int_0^1 (1 - x^2)\, dx = 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left[1 - \frac{1}{3}\right] = 2\left[\frac{2}{3}\right]$$

$$a_0 = \frac{4}{3}$$

To find a_n:

$$a_n = 2\int_0^1 (1 - x^2)\cos n\pi x \;\; dx$$

$$= 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left(-\frac{2(-1)^n}{n^2\pi^2}\right)$$

$$a_n = -\frac{4(-1)^n}{n^2\pi^2}$$

Required Fourier series is

$$\text{f(x)} = \frac{\left(4/3\right)}{2} + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2\pi^2}\cos n\pi x$$

Result: $\text{f(x)} = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi x$

[A.U. M/J 2016]

26. Find the half range sine series of $f(x) = x\cos\pi x \quad in \quad (0,1).$

Solution: Given $f(x) = x\cos\pi x$ in $(0,1)$.

Formula: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{1}\int_0^1 x\cos\pi x\,\sin\frac{n\pi x}{1}\, dx$$

$$= 2\int_0^1 x\cos\pi x\sin n\pi x\,dx$$

$$= 2\int_0^1 x\ \sin n\pi x\cos\pi x\,dx$$

$$= \int_0^1 x\ (2\sin n\pi x\cos\pi x)dx$$

$$= \int_0^1 x\ [\sin(n+1)\pi x + \sin(n-1)\pi x\,]dx$$

By applying Bernoulli's formula

$$u = x \quad ; \quad \int dv = \sin(n+1)\pi x + \sin(n-1)\pi x$$

$$u' = 1 \quad ; \quad v = -\frac{\cos(n+1)\pi x}{n+1} - \frac{\cos(n-1)\pi x}{n-1}$$

$$u'' = 0 \quad ; \quad v_1 = -\frac{\sin(n+1)\pi x}{(n+1)^2} - \frac{\sin(n-1)\pi x}{(n-1)^2}$$

$$\int uv\,dx = uv - u'v_1 + u''v_2 - \ldots$$

$$= \left[-x\left(\frac{\cos(n+1)\pi x}{(n+1)} + \frac{\cos(n-1)\pi x}{(n-1)}\right) + \left(\frac{\sin(n+1)\pi x}{(n+1)} + \frac{\sin(n-1)\pi x}{(n-1)}\right)\right]_0^1$$

$$= \left[-1\left(\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)}\right) + \left(\frac{\sin(n+1)\pi}{(n+1)} + \frac{\sin(n-1)\pi}{(n-1)}\right)\right]$$

$$= \left[(-1)\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right) + (0+0)\right]$$

$$= (-1)(-1)^{n+1}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)$$

$$= (-1)^{n+2}\left(\frac{n-1+n+1}{n^2-1}\right)$$

$$b_n = (-1)^n\left(\frac{2n}{n^2-1}\right)$$

$put\quad n = 1$

$$b_1 = 2\int_0^1 x\cos\pi x\sin\pi x\,dx$$

$$= \int_0^1 x\ (2\sin \pi x \cos \pi x) dx$$

$$= \int_0^1 x \sin 2\pi x\ dx$$

By applying Bernoulli's formula

$$u = x \quad ; \quad \int dv = \sin 2\pi x$$

$$u' = 1 \quad ; \quad v = -\frac{\cos 2\pi x}{2}$$

$$u'' = 0 \quad ; \quad v_1 = -\frac{\sin 2\pi x}{4}$$

$$\int uv\ dx = uv - u'v_1 + u''v_2 - \ldots$$

$$= \left[-x\frac{\cos 2\pi\, x}{2} + \frac{\sin 2\pi\, x}{4} \right]_0^1$$

$$= \left[(-1)\frac{\cos 2\pi(\ )}{2} + \frac{\sin 2\pi}{4} \right] - (0+0)$$

$$= \frac{(-1)(1)}{2}$$

$$b_1 = \frac{-1}{2}$$

$$f(x) = \sum_{n=1}^{\infty} b_n\ \sin\frac{n\pi\, x}{l}$$

$$= b_1 + \sum_{n=2}^{\infty} b_n\ \sin\frac{n\pi\, x}{l}$$

Result : $$f(x) = \frac{-1}{2} + \sum_{n=2}^{\infty} (-1)^n \left(\frac{2n}{n^2-1} \right) \sin n\pi x$$

PROBLEMS IN COMPLEX FORM

FORMULA $$f(x) = \sum_{n=-\infty}^{\infty} C_n\ e^{\frac{in\pi x}{l}}$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\ e^{\frac{-in\pi x}{l}} dx$$

27. Find the complex form of the Fourier series of $f(x) = e^{-x}\ in\ -1 < x \le 1$

Solution: **[A.U. N/D 2009, A/M 2015]**

Given $f(x) = e^{-x}\ in\ -1 < x \le 1$

Formula: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}}\, dx$$

Here $l = 1,$

$$\therefore \quad f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{in\pi x}$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\, e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-[1+in\pi]x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-[1+in\pi]x}}{-[1+in\pi]}\right]_{-1}^{1}$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-[1+in\pi]x}\right]_{-1}^{1}$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-1}\, e^{-in\pi} - e^{1}\, e^{in\pi}\right]$$

$$= \frac{-1}{2(1+in\pi)}\left[e^{-1}(-1)^n - e^{1}(-1)^n\right]$$

$$= \frac{-(-1)^n}{(1+in\pi)}\left[\frac{e^{-1}-e^{1}}{2}\right]$$

$$C_n = \frac{(-1)^n}{(1+in\pi)}\left[\sinh 1\right]$$

$$f(x) = \sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n}{1+in\pi}\sinh 1\right] e^{in\pi x}$$

Result: $f(x) = \sinh 1 \sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n(1-in\pi)}{1+n^2\pi^2}\right] e^{in\pi x}$

28. Find the complex form of Fourier series for the function $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$ Where a is a real constant. Hence, deduce that $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2} = \frac{\pi}{a \sinh a\pi}$.

Solution: **Formula:** $f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{inx}$ **[A.U.N/D 2015]**

Where $C_n = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(x)\, e^{-inx}\ dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} e^{-inx}\ dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\ dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{(a-in)x}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}\ e^{-in\pi} - e^{-a\pi} e^{in\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}\ (-1)^n - e^{-a\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{\pi(a-in)}\left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right] \qquad \because \sin h\theta = \left[\frac{e^{\theta} - e^{-\theta}}{2}\right]$$

$$C_n = \frac{(-1)^n(a+in)}{\pi(a^2+n^2)}\sin h\, a\pi$$

$$f(x) = \frac{(-1)^n(a+in)}{\pi(a^2+n^2)}\sin h\, a\pi\ e^{inx}$$

Result: *i.e.,* $e^{ax} = \frac{\sin h\, a\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2} e^{inx}$

29.Expand $f(x) = e^{-ax}$ in the interval $(-\pi,\pi)$ as a complex form of Fourier series

Solution:

Given $f(x) = e^{-ax}$

Formula: $f(x) = \sum_{n=-\infty}^{\infty} C_n\ e^{inx}$ **[A.U.N/D 2016]**

Where $C_n = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f(x)\, e^{-inx}\ dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ax} e^{-inx}\ dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(-a-in)x}\ dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(-a-in)x}}{-a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(-a-in)}\left[e^{(-a-in)x}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(-a-in)}\left[e^{(-a-in)\pi} - e^{-(-a-in)\pi}\right]$$

$$= \frac{1}{2\pi(-a-in)}\left[e^{-a\pi}\ e^{-in\pi} - e^{a\pi}e^{in\pi}\right]$$

$$= \frac{1}{2\pi(-a-in)}\left[e^{-a\pi}\ (-1)^n - e^{a\pi}(-1)^n\right]$$

$$= \frac{(-1)^{n+1}}{\pi(-a-in)}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right] \qquad \because \sin h\theta = \left[\frac{e^{\theta}-e^{-\theta}}{2}\right]$$

$$C_n = \frac{(-1)^{n+1}(-a+in)}{\pi\left(a^2+n^2\right)}\sin h\,a\pi$$

$$f(x) = \frac{(-1)^{n+1}(-a+in)}{\pi\left(a^2+n^2\right)}\sin h\,a\pi\ e^{inx}$$

Result: $\quad$ *i.e.,* $e^{-ax} = \dfrac{\sin h\,a\pi}{\pi}\displaystyle\sum_{n=-\infty}^{\infty}(-1)^{n+1}\frac{-a+in}{a^2+n^2}e^{inx}$

30. Find the complex form of Fourier series $f(x) = e^{-ax}$ in the interval $-l < x < l$

Solution:

Given $f(x) = e^{-ax} \quad -l < x < l$

Formula: $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} C_n\ e^{\frac{in\pi x}{l}}$ **[A.U.A/M 2017]**

Where $C_n = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x)\mathrm{e}^{\frac{-in\pi x}{l}}\ dx$

$$= \frac{1}{2l}\int_{-l}^{l} e^{-ax}e^{-\frac{in\pi x}{l}}\ dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{-\left[\frac{al+in\pi}{l}\right]x}\ dx$$

$$= \frac{1}{2l}\left[\frac{e^{\frac{(-al-in)x}{l}}}{\frac{-al-in}{l}}\right]_{-l}^{l}$$

$$= \frac{1}{-2(al+in\pi)}\left[e^{-(al+in\pi)} - e^{-(al+in\pi)(-1)}\right]$$

$$= \frac{1}{-2(al+in\pi)}\left[e^{-al}\ e^{-in\pi} - e^{al}e^{in\pi}\right]$$

$$= \frac{-1}{2(al+in\pi)}\left[e^{-al}\ (-1)^n - e^{al}(-1)^n\right]$$

$$= \frac{(-1)^n}{(al+in\pi)}\sinh al \qquad \because \sin\ h\theta = \left[\frac{e^{\theta} - e^{-\theta}}{2}\right]$$

$$C_n = \frac{(-1)^n (al - in\pi)}{((al)^2 + (n\pi)^2)}\sin h\, a\pi$$

$$i.e.,\ f(\text{x}) = \sin h\, al \sum_{n=-\infty}^{\infty} (-1)^n \frac{al - in\pi}{(al)^2 + (n\pi)^2} e^{\frac{in\pi x}{l}}$$

Result: $$i.e.,\ e^{-ax} = \sin h\, al \sum_{n=-\infty}^{\infty} (-1)^n \frac{al - in\pi}{(al)^2 + (n\pi)^2} e^{\frac{in\pi x}{l}}$$

31. Find the complex form of the Fourier series $f(x) = \cos ax \quad in\ -\pi < x < \pi$ **.**

Solution: **[A.U. MAY 2013]**

Given the Complex form of Fourier series of $f(x) = \cos ax \quad in\ -\pi < x < \pi$ is given by

Formula: $$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \text{.....................(1)}$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos axe^{-inx}dx$$

$$= \frac{1}{2\pi}\left\{\frac{e^{-inx}}{i^2n^2 + a^2}(-in\cos ax + a\sin ax)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left\{\frac{e^{-in\pi}}{a^2 - n^2}(-in\cos ax + a\sin ax) - \frac{e^{in\pi}}{a^2 - n^2}(-in\cos ax - a\sin ax)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2 - n^2)}\left[-in\cos a\pi(e^{-in\pi} - e^{in\pi}) + a\sin a\pi(e^{in\pi} + e^{-in\pi})\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[2in\cos a\pi \sin n\pi + 2a\sin a\pi \cos n\pi\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[2a\sin a\pi \cos n\pi\right] \qquad [\because \sin n\pi = 0]$$

$$= \frac{(-1)^n a\sin a\pi}{\pi(a^2-n^2)} \text{.........................}(2)$$

Substituting (2) in (1) we get,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin a\pi.a}{\pi\left(a^2-n^2\right)}.e^{inx}$$

Result: $$\cos ax = \frac{\sin a\pi.a}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\pi\left(a^2-n^2\right)}.e^{inx}$$

HARMONIC ANALYSIS

FORMULA: (T-FORM)

$$f(x) = f(\theta) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + a_3\cos 3\theta + b_3\sin 3\theta + ...$$

$$a_0 = 2\left[\frac{\sum y}{n}\right],\ a_1 = 2\left[\frac{\sum y\cos\theta}{n}\right], b_1 = 2\left[\frac{\sum y\sin\theta}{n}\right], a_2 = 2\left[\frac{\sum y\cos 2x}{n}\right],$$

$$b_2 = 2\left[\frac{\sum y\sin 2x}{n}\right]$$

32. Compute upto first harmonics of the fourier series of f(x) given by the following table:

x	**0**	**T/6**	**T/3**	**T/2**	**2T/3**	**5T/6**	**T**
f(x)	**1.98**	**1.30**	**1.05**	**1.30**	**-0.88**	**-0.25**	**1.98**

Solution: **[A.U NOV 2009]**

x	$\theta = \frac{2\pi x}{T}$	y	$y\cos\theta$	$y\sin\theta$
0	$0 = 0^0$	1.98	1.98	0
$T/6$	$\pi/3 = 60^0$	1.30	0.65	1.1258
$T/3$	$2\pi/3 = 120^0$	1.05	-0.525	0.9093
$T/2$	$\pi = 180^0$	1.30	-1.3	0
$2T/3$	$4\pi/3 = 240^0$	-0.88	0.44	0.762
$5T/6$	$5\pi/3 = 300^0$	-0.25	-0.125	0.2165
		$\sum y = 4.5$	$\sum y\cos\theta = 1.12$	$\sum y\sin\theta = 3.0136$

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{4.5}{6}\right] = 1.5$$

$$a_1 = 2\left[\frac{\sum y\cos\theta}{n}\right] = 2\left[\frac{1.12}{6}\right] = 0.3733$$

$$b_1 = 2\left[\frac{\sum y\sin\theta}{n}\right] = 2\left[\frac{3.0136}{6}\right] = 1.0045$$

Formula: $y = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$

$$= \frac{1.5}{2} + 0.373\cos\theta + 1.0045\sin\theta$$

Result: $y = 0.75 + 0.373\cos\theta + 1.0045\sin\theta \quad where \quad \theta = \frac{2\pi x}{T}$

33. Determine the first two harmonics of Fourier series for the following data:

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
$f(x)$:	1.98	1.30	1.05	1.30	-0.88	-0.25

Solution: **[A.U.N/D 2015]**

x	y	y cos x	y sin x	y cos 2x	y sin 2x
0^0	1.98	1.98	0.000	1.98	0.000
$\pi/3 = 60^0$	1.30	0.65	1.126	-0.65	1.126
$2\pi/3 = 120^0$	1.05	-0.525	0.909	-0.525	-0.909
$\pi = 180^0$	1.30	-1.30	0.000	1.30	0.000
$4\pi/3 = 240^0$	-0.88	0.44	0.762	0.44	-0.762
$5\pi/3 = 300^0$	-0.25	-0.125	0.217	0.125	0.2167
	$\sum y = 4.5$	$\sum y\cos x = 1.12$	$\sum y\sin x = 3.014$	$\sum y\cos 2x = 2.67$	$\sum y\sin 2x = -0.328$

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{4.5}{6}\right] = 1.5$$

$$a_1 = 2\left[\frac{\sum y\cos x}{n}\right] = 2\left[\frac{1.12}{6}\right] = 0.373$$

$$a_2 = 2\left[\frac{\sum y\cos 2x}{n}\right] = 2\left[\frac{2.67}{6}\right] = 0.89$$

$$b_1 = 2\left[\frac{\sum y\sin x}{n}\right] = 2\left[\frac{3.014}{6}\right] = 1.005$$

$$b_2 = 2\left[\frac{\sum y\sin 2x}{n}\right] = 2\left[\frac{-0.328}{6}\right] = -0.109$$

Formula: $y = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$

$$= \frac{1.5}{2} + (0.373\cos x + 1.005\sin x) + (0.89\cos 2x - 0.109\sin 2x)$$

Result: $y = 0.75 + (0.373\cos x + 1.005\sin x) + (0.89\cos 2x - 0.109\sin 2x)$

HARMONIC ANALYSIS

FORMULA: (L-FORM)

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right) + \left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right) + ...$$

$$a_0 = \frac{2}{n}\sum y\ , \qquad a_1 = \frac{2}{n}\sum y\cos\frac{\pi x}{l}, \qquad a_2 = \frac{2}{n}\sum y\cos\frac{2\pi x}{l}$$

$$b_1 = \frac{2}{n}\sum y\sin\frac{\pi x}{l}, \qquad b_2 = \frac{2}{n}\sum y\sin\frac{2\pi x}{l}$$

34. Find the Fourier series as far as the second harmonic to represent the function f(x) with period given in the following table. [A.U DEC 2010, 2012, N/D 2016, A/M 17]

X	0	1	2	3	4	5
f(x)	9	18	24	28	26	20

Solution:

Here the length of the interval is $2l = 6, \quad l = 3$

The required Fourier series is

Formula: $y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right) + \left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right) + ...$

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$\cos\frac{2\pi x}{3}$	$\sin\frac{2\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	9	1	0	1	0	9	0	9	0
1	18	0.5	0.866	-0.5	0.866	9	15.588	-9	15.588
2	24	-0.5	0.866	-0.5	-0.866	-12	20.785	-12	-20.785
3	28	-1	0	1	0	-28	0	28	0
4	26	-0.5	-0.866	-0.5	0.866	-13	-22.517	-13	22.517
5	20	0.5	-0.866	-0.5	-0.866	10	-17.321	-10	-17.321
$\sum$	125					-25	-3.465	-7	0.0004

$$y = \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3} + a_2\cos\frac{2\pi x}{3} + b_2\sin\frac{2\pi x}{3}$$

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{n}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}(-25) = -8.33$$
$$a_2 = \frac{2}{n}\sum y\cos\frac{2\pi x}{3} = \frac{2}{6}(-7) = -2.333$$
$$b_1 = \frac{2}{n}\sum y\sin\frac{\pi x}{3} = \frac{2}{6}(-3.465) = -1.16$$
$$b_2 = \frac{2}{n}\sum y\sin\frac{2\pi x}{3} = \frac{2}{6}(0.0004) = 0.00013$$
$$y = \frac{41.67}{2} - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3} - 2.333\cos\frac{2\pi x}{3} + 0.00013\sin\frac{2\pi x}{3}$$

Result: $y = 20.84 - 8.33\cos\frac{\pi x}{3} - 1.16\sin\frac{\pi x}{3} - 2.333\cos\frac{2\pi x}{3} + 0.00013\sin\frac{2\pi x}{3}$

HARMONIC ANALYSIS

FORMULA: (2π-FORM)

$$y = \frac{a_0}{2} + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x$$
$$a_0 = \frac{2}{n}\sum y,\quad a_1 = \frac{2}{n}\sum y\cos x,\quad a_2 = \frac{2}{n}\sum y\cos 2x$$
$$b_1 = \frac{2}{n}\sum y\sin x,\quad b_2 = \frac{2}{n}\sum y\sin 2x$$

35. Find the Fourier series up to second harmonic for y = f(x) from the following values

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1

Solution: **[A.U NOV 2013, 2014,N/D 2017, A/M 2018]**

Formula: $y = \frac{a_0}{2} + \left(a_1\cos x + b_1\sin x\right) + \left(a_2\cos 2x + b_2\sin 2x\right)$

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	1	1	0	1	0	1	0	1	0
$\frac{\pi}{3}$	1.4	0.5	0.866	-0.5	0.866	0.7	1.2124	-0.7	1.2124
$\frac{2\pi}{3}$	1.9	-0.5	0.866	-0.5	-0.866	-0.95	1.6454	-0.95	-1.6454
π	1.7	-1	0	1	0	-1.7	0	1.7	0
$\frac{4\pi}{3}$	1.5	-0.5	-0.866	-0.5	0.866	-0.75	-1.299	-0.75	1.299
$\frac{5\pi}{3}$	1.2	0.5	-0.866	-0.5	-0.866	0.6	-1.0392	-0.6	-1.0392
	8.7					-1.1	0.5196	-0.3	-0.1732

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(8.7) = 2.9$$

$$a_1 = \frac{2}{n}\sum y\cos x = \frac{2}{6}(-1.1) = -0.37$$

$$a_2 = \frac{2}{n}\sum y\cos 2x = \frac{2}{6}(-0.3) = -0.1$$

$$b_1 = \frac{2}{n}\sum y\sin x = \frac{2}{6}(0.5196) = 0.17$$

$$b_2 = \frac{2}{n}\sum y\sin 2x = \frac{2}{6}(-0.1732) = -0.06$$

$$y = \frac{2.9}{2} - 0.37\cos x + 0.17\sin x - 0.1\cos 2x - 0.06\sin 2x$$

$$y = 1.45 - 0.37\cos x + 0.17\sin x - 0.1\cos 2x - 0.06\sin 2x$$

36. Find the Fourier cosine series up to third harmonic to represent the function given by the following data:

X	0	1	2	3	4	5
f(x)	4	8	15	7	6	2

Solution: **[A.U. M/J 2016]**

Here the length of the interval is 2l=6, i.e., l=3

The required Fourier series is

Formula:

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}\right) + \left(a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l}\right) + \left(a_3\cos\frac{3\pi x}{l} + b_2\sin\frac{3\pi x}{l}\right)$$

$$y = \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3} + a_2\cos\frac{2\pi x}{3} + b_2\sin\frac{2\pi x}{3} + a_3\cos\frac{3\pi x}{3} + b_3\sin\frac{3\pi x}{3}$$

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$\cos\frac{2\pi x}{3}$	$\sin\frac{2\pi x}{3}$	$\cos\frac{3\pi}{3}$	$\sin\frac{3\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$	$y\cos\frac{3\pi x}{3}$	$y\sin\frac{3\pi x}{3}$
0	4	1	0	1	0	1	0	4	4	4	0	0	0
1	8	0.5	0.866	-0.5	0.866	-1	0	4	-4	-8	6.928	6.928	0
2	15	-0.5	0.866	-0.5	-0.866	1	0	-7.5	-7.5	15	-12.99	-12.99	0
3	7	-1	0	1	0	-1	0	-7	7	-7	0	0	0
4	6	-0.5	-0.866	-0.5	0.866	1	0	-3	-3	6	5.196	5.196	0
5	2	0.5	-0.866	-0.5	-0.866	-1	0	1	-1	-2	-1.732	-1.732	0
$\sum$	42							-8.5	-4.5	8	-2.598	-2.598	0

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(42) = 14$$

$$a_1 = \frac{2}{n}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}(-8.5) = -2.833$$

$$a_2 = \frac{2}{n}\sum y\cos\frac{2\pi x}{3} = \frac{2}{6}(-4.5) = -1.5$$

$$a_3 = \frac{2}{n}\sum y\cos\frac{3\pi x}{3} = \frac{2}{6}(8) = 2.667$$

$$b_1 = \frac{2}{n}\sum y\sin\frac{\pi x}{3} = \frac{2}{6}(-2.598) = -0.866$$

$$b_2 = \frac{2}{n}\sum y\sin\frac{2\pi x}{3} = \frac{2}{6}(-2.598) = -0.866$$

$$b_3 = \frac{2}{n}\sum y\sin\frac{3\pi x}{3} = \frac{2}{6}(0) = 0$$

$$y = \frac{14}{2} - 2.833\cos\frac{\pi x}{3} - 0.866\sin\frac{\pi x}{3} - 1.5\cos\frac{2\pi x}{3} - 0.866\sin\frac{2\pi x}{3} + 2.667\cos\frac{3\pi x}{3}$$

$$y = 7 - 2.833\cos\frac{\pi x}{3} - 0.866\sin\frac{\pi x}{3} - 1.5\cos\frac{2\pi x}{3} - 0.866\sin\frac{2\pi x}{3} + 2.667\cos\frac{3\pi x}{3}$$

ANNA UNIVERSITY QUESTIONS

1. Find the Fourier series expansion of f(x) = $\begin{cases} x \ for\ 0 < x \le \pi \\ 2\pi - x \ for\ \pi \le x < 2\pi \end{cases}$ Also, deduce that

$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+.........=\frac{\pi^2}{8}$ [Pg: No: 18]

2. Find the Fourier series for f(x) $=\begin{cases} 1 \ for\ 0 < x < \pi \\ 2 \ for\ \pi < x < 2\pi \end{cases}$ [Pg: No: 20] [A.U.N/D 2013]

3. Find the Fourier Series of period 2π for the function $f(x) = x\cos x$ in $0 < x < 2\pi$ [Pg. No: 21] [A.U. A/M 2017]

4. Find the Fourier series of the periodic function defined by $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

$$Deduce\ that\ \frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{1^2}+...+\infty=\frac{\pi^2}{8}$$ [Pg: No:23] [A.U Nov 2009]

5. Expand the function $f(x) = x\sin x$ as a Fourier series in the interval $(-\pi, \pi)$. [Pg: No:25] [A.U. Nov 2000]

6. Find the Fourier series of $f(x) = x \ in \ -\pi < x < \pi.$ [Pg: No:26] [A.U. M/J 2016]

7. Find the Fourier series for $\mathrm{f(x)} = |\cos x|$ in the interval $(-\pi, \pi)$. [Pg: No: 27] [A.U M/J 2016]

8. Find the Fourier series expansion of $f(x) = x + x^2 \ in \ (-\pi, \pi)$ and hence deduce the value of $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...$ [Pg: No:29] [A.U. DEC. 2017]

9. Find the Fourier series for $f(x) = x^2 \ in \ -\pi < x < \pi.$ Hence show that

$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+...=\frac{\pi^4}{90}$ [Pg: No: 31] [A.U.MAY 2013,2014,2018]

10 . If $f(x) = \begin{cases} 1-x, -\pi \le x \le 0 \\ 1+x, 0 \le x \le \pi \end{cases}$ find the Fourier series of f(x) and hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+......=\frac{\pi^2}{8}$ [Pg: No: 32] [A.U. MAY 2011, NOV 2013]

11. Find the Fourier series of $f(x) = |\sin x| \ in -\pi < x < \pi$ of periodicity 2π . [Pg: No: 33] [AU A/M 2015]

12. Find the Fourier series expansion the following periodic function of period 4

$f(x) = \begin{cases} 2+x & -2 \le x \le 0 \\ 2-x & 0 < x \le 2 \end{cases}$. Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...=\frac{\pi^2}{8}$. [Pg: No: 35] [A.U. N/D 2015]

13. Find the half-range cosine series of $f(x) = (\pi - x)^2$ in (0, π). Hence find the sum of the Series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots\ldots\ldots\infty$ [Pg: No: 37] [A.U. N/D 2009,2015]

14. Find the half-range cosine series of the function $f(x) = x(\pi - x)$ in the interval $0 < x < \pi$. Hence deduce that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots = \frac{\pi^4}{90}$. [Pg: No: 38] [A.U. M/J 2010, A/M 2018]

15. Obtaion the half-range cosine series for $f(x) = x \quad in \quad (0,\pi)$ and hence Prove that $\frac{1}{1^2}+\frac{1}{3^2}+\ldots$ [Pg: No: 40][A.U. N/D 2007, 2012,N/D 2017]

16. Expand f(x) = $\begin{cases} x & for\ 0 < x \le 1 \\ 2-x & for\ 1 \le x < 2 \end{cases}$ as a series of cosines in the interval (0,2).
[Pg. No. 42][A.U. N/D 2016, A/M 2017]

17. Find the half range cosine series of f(x) = x in 0 < x< π.
[Pg: No: 43] [A.U. N/D 2010]

18. Find the half range sine series of $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$. Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$. [Pg: No: 43] [AU A/M 2015]

19. Find the half range sine series $f(x) = lx - x^2 \ in\ (0,l)$ [Pg: No: 44] [AU Nov 2013]

20. Find the half-range cosine series for $f(x) = (x-1)^2 \ in\ 0 < x < 1$.
[Pg: No: 45] [A.U.M/J 2013, N/D 2014]

21. Obtain fourier cosine series of $f(x) = x$, $0<x<4$ hence deduct that $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots+\infty$ [Pg: No: 46] [AU N/D 2014]

22. Obtain the Fourier series for the function f(x)= $\begin{cases} x & for\ 0 \le x \le 1 \\ 2-x & for\ 1 \le x \le 2 \end{cases}$

(or) Expand f(x)= $\begin{cases} x & for\ 0 \le x \le 1 \\ 2-x & for\ 1 \le x \le 2 \end{cases}$ as a series of cosine in the interval (0,2)
[Pg. No: 47] [A.U MAY 2011, 2013,N/D 2016, A/M 17]

23. Obtain the Fourier series for the function $f(x)=\begin{cases} l-x & for\ 0\le x\le l \\ 0 & for\ l\le x\le 2l \end{cases}$

[Pg. No: 49] [A.U N/D 17]

24. Obtain the sine series for the function $f(x)=\begin{cases} x\ \ for\ 0\le x\le l/2 \\ l-x\ \ for\ l/2\le x\le l \end{cases}$

[Pg:No:50] [A.U. May 2011, 2013]

25. Find the Fourier series expansion of $f(x)=1-x^2$ in the interval (-1,1)

[Pg: No: 51] [A.U DEC 2010]

26. Find the half range sine series of $f(x)=x\cos\pi x\ \ in\ \ (0,1)$. [Pg: No: 52][A.U.M/J 2016]

27. Find the complex form of the Fourier series of $f(x)=e^{-x}\ in\ -1<x\le 1$

[Pg: No: 54] [A.U. N/D 2009, A/M 2015]

28. Find the complex form of Fourier series for the function $f(x)=e^{ax}$ in the interval $(-\pi,\pi)$ Where a is a real constant. Hence, deduce that $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{\pi}{a\sinh a\pi}$.

[Pg: No: 55] [A.U. N/D 2015]

29.Expand $f(x)=e^{-ax}$ in the interval $(-\pi,\pi)$ as a complex form of Fourier series

[Pg: No: 56] [A.U. N/D 2016]

30. Find the complex form of Fourier series $f(x)=e^{-ax}$ in the interval $-l<x<l$

[Pg: No: 57] [A.U. A/M 2017]

31. Find the complex form of the Fourier series $f(x)=\cos ax\ \ in\ -\pi<x<\pi$.

[Pg: No:58] [A.U. M/J 2013]

32. Compute upto first harmonics of the fourier series of f(x) given by the following table:

x	0	T/6	T/3	T/2	2T/3	5T/6	T
f(x)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

[Pg: No: 59][A.U. N/D 2009]

33. Determine the first two harmonics of Fourier series for the following data:

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
$f(x)$:	1.98	1.30	1.05	1.30	-0.88	-0.25

[Pg: No: 60] [A.U. N/D 2015]

34. Find the Fourier series as far as the second harmonic to represent the function f(x) with period given in the following table.

X	0	1	2	3	4	5
f(x)	9	18	24	28	26	20

[Pg: No: 61][A.U. N/D 2010,2012, 16, 17]

35. Find the Fourier series up to second harmonic for y = f(x) from the following values

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1

[Pg: No: 62] [A.U MAY 2015, NOV 2013, 2014,2017, A/M 2018]

36. Find the Fourier cosine series up to third harmonic to represent the function given by the following data:

X	0	1	2	3	4	5
f(x)	4	8	15	7	6	2

[Pg: No: 63] [A.U. M/J 2016]

IMPORTANT QUESTIONS

PART-A

1. State the sufficient condition for $f(x)$ to be expressed as a Fourier series. [A.U NOV. 2009],

4. Find the Root mean square value of $f(x) = x^2$ in $(0\ ,\ l\ )$ [A.U.NOV. 2010]

5. Give the expression for the Fourier series coefficient b_n for the function $f(x)$ defined in $(-2,2)$. [A.U MAY 2011]

6. Without finding the values of a_0, a_n, b_n the Fourier coefficients of Fourier series, for the function $f(x) = x^2$ in the interval $(0,\pi)$ Find the value of $\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$. [A.U MAY 2011]

7. Find the RMS value of the function $f(x) = x$ in $(0,l)$. [A.U.DEC 2011]

8. Define RMS value of a function $f(x)$ over the interval (a,b). [A.U DEC 2012]

9. Determine the value of a_n in the Fourier series expansion of $f(x) = x^3$ in $(-\pi,\pi)$. [A.U MAY 2008]

10. Let $f(x)$ be defined in the interval $(0,2\pi)$ by $f(x) = \begin{cases} \frac{1+\cos x}{\pi - x}, & 0 < x < \pi \\ \cos x, & \pi < x < 2\pi \end{cases}$

$f(x+2\pi) = f(x)$. Find the value of $f(\pi)$. [A.U MAY 2009]

11. In the Fourier series expansion of

$$f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi < x < 0 \\ 1-\frac{2x}{\pi}, & 0 < x < \pi \end{cases} \quad in\ (-\pi,\pi)$$ Find the value of b_n. [A.U MAY 2005]

12. Write down the complex form of the Fourier series for $f(x)\ in\ (c, c+2\pi)$. [A.U DEC 2010]

13. State the Parseval's theorem in Fourier series. [A.U MAY 2011]

14. If the Fourier series of $f(x)=\begin{cases}0 & for -\pi \le x \le 0\\ \sin x & for\ 0 \le x \le \pi\end{cases}$ is

$$f(x)=\frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^{\infty}\cos\frac{2nx}{4n^2-1}+\frac{1}{2}\sin x$$. Find b_n. [A.U MAY 2011]

15. Find the RMS value of $f(x)=1-x$ in $0<x<1$ [A.U MAY 2010]

16. If $f(x)$ is an odd function in the interval $(-1,1)$, what are the values of a_0 and a_n. [A.U MAY 2010]

17. Check whether $f(x)=1+x+x^2$ is odd or even.

18. Find the coefficient a_8 of $\cos 8x$ in the F.C.T of the function $f(x)=\sin 8x$ in $(0,\pi)$.

19. If $x=2\left\{\frac{\sin x}{1}-\frac{\sin 2x}{2}+..........\right\}$ in $0<x<\pi$. Prove that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

20. Find the value of a_0 in the fourier series expansion of f(x) = e^x in $(0,2\pi)$. [AU Nov 2013]

21. In the expansion of $f(x)=\sinh x\ in\ (-\pi,\pi)$ as a Fourier series, find the coefficient of b_n

22. Find the Fourier constants b_n for $x\sin x$ in $(-\pi,\pi)$. [A.U DEC 2013]

23. State Parseval's Identity for the half range cosine expansion of $f(x)$ in $(0,1)$.

24. State whether $y=\tan x$ can be expanded as a Fourier series. If so how ?If not why?

25. Write a_0,a_n in the expansion of $x+x^3$ as a Fourier series in $(-\pi,\pi)$. [A.U DEC 2010]

26. Find the sine series of function $f(x)=1,\quad 0\le x\le\pi$. [A.U N/D 2015]

27. Find the half range sine series expansion of f(x) = 1 in (0, 2) [A.U Nov 2013]

28. What do you mean by Harmonic analysis? [A.U MAY 2010,2013]

29. If the Fourier series for the function $f(x)=\begin{cases}0 & if\ 0<x<\pi\\ \sin x & if\ \pi<x<2\pi\end{cases}$

$$f(x)=-\frac{1}{\pi}+\frac{2}{\pi}\left[\frac{\cos 2x}{1.3}+\frac{\cos 4x}{3.5}+\frac{\cos 6x}{5.7}+......\right]+\frac{1}{2}\sin x$$

Deduce that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-......=\frac{\pi-2}{4}$ [A.U MAY 2004]

30. Suppose the function $x\cos x$ has the series expansion $\sum_{n=1}^{\infty}b_n\sin nx$ find the value of $b_1\ in\ (-\pi,\pi)$.

31. If $f(x)$ is an odd function of x in (-2, 2), what are the value of a_0,a_n?

32. Find b_n in the expansion x^4 of as a Fourier series in $(-\pi,\pi)$.

33. If $(\pi-x)^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ in $0<x<2\pi$ then deduct that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ [AU NOV/DEC 2014]

34 . If the fourier series of the function $f(x) = x$, $-\pi < x < x$ with period 2π is given by $f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right)$ then find the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

[AU A/M 2015]

35. The instantaneous current "i" at time t of an alternating current wave is given by $i = I_1 \sin(\omega t + \alpha_1) + I_3 \sin(\omega t + \alpha_3) + I_5 \sin(\omega t + \alpha_5) + ...$ find the effective value of the current "i" [A.U A/M 2015]

36. Find the root mean square value of $f(x) = x(l - x)$ in $0 \le x \le l.$ [A.U. N/D 2015]

37. Find the value of the Fourier series of $f(x) = \begin{cases} 0 & in\ (-c,0) \\ 1 & in\ (0,c) \end{cases}$ at the point of discontinuity x=0. [A.U. M/J 2016]

38. Find the value of b_n in the Fourier series expansion of $f(x) = \begin{cases} x+\pi & in\ (-\pi,0) \\ -x+\pi & in\ (0,\pi) \end{cases}$

[A.U. M/J 2016]

39. What is the sum of the Fourier series at a point $x = x_o$ where the function $f(x)$ has a finite discontinuity .

40. Obtain the first term of the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$

[A.U NOV. 2009]

PART-B

1. (a). Find the Fourier series expansion of f(x) = $\begin{cases} x \ for\ 0 < x \le \pi \\ 2\pi - x \ for\ \pi \le x < 2\pi \end{cases}$ Also, deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$$

(b). Find the Fourier series for f(x) $= \begin{cases} 1 \ for\ 0 < x < \pi \\ 2 \ for\ \pi < x < 2\pi \end{cases}$ [AU Nov 2013]

2. (a). Find the Fourier series of the periodic function defined by $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

$$Deduce\ that \frac{1}{1^2} + \frac{1}{1^2} + \frac{1}{1^2} + ... + \infty = \frac{\pi^2}{8}$$ [A.U NOV 2009]

(b). Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $(-\pi, \pi)$.

[A.U. Nov 2000]

3. (a). Find the Fourier series of $f(x) = x \ in \ -\pi < x < \pi.$ [A.U. M/J 2016]

(b). Find the Fourier series for $\text{f(x)} = |\cos x|$ in the interval $(-\pi, \pi)$.

[A.UM/J 2016]

4. (a). Find the Fourier series expansion of $f(x) = x + x^2 \ in \ (-\pi, \pi)$ and hence deduce the value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$ [A.U. DEC. 2017]

(b). Find the Fourier series for $f(x) = x^2 \ in \ -\pi < x < \pi.$ Hence show that

$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+...=\frac{\pi^4}{90}$ [A.U., 2013, Nov 2014,2018]

5. (a). If $f(x)=\begin{cases}1-x,-\pi\le x\le 0\\ 1+x,0\le x\le\pi\end{cases}$ find the Fourier series of f(x) and hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+......=\frac{\pi^2}{8}$

[A.U. MAY 2011, NOV 2013]

(b). Find the Fourier series of $f(x)=|\sin x|\ in -\pi<x<\pi$ of periodicity 2π.

[AU A/M 2015]

6.(a). Find the Fourier series expansion the following periodic function of period 4 $f(x)=\begin{cases}2+x & -2\le x\le 0\\ 2-x & 0<x\le 2\end{cases}$. Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...=\frac{\pi^2}{8}$.

[A.U. N/D 2015]

(b). Find the half-range cosine series of $f(x)=(\pi-x)^2$ in (0, π). Hence find the sum of the Series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.............\infty$

[A.U NOV 2009 , N/D 2015]

7. (a).Find the half-range cosine series of the function $f(x)=x(\pi-x)$ in the interval $0<x<\pi$. Hence deduce that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...=\frac{\pi^4}{90}$. [A.U. May 2018]

(b). Obtaion the half-range cosine series for $f(x)=x \quad in \quad (0,\pi)$ and hence Prove that $1+\frac{1}{2^4}+\frac{1}{3^4}+...=\frac{\pi^4}{96}$ [A.U Nov 2007, 2012,2017]

8. (a). Find the half range cosine series of f(x) = x in 0 < x< π. [A.U DEC 2010]

(b). Find the half range sine series of $f(x)=\begin{cases}x & 0<x<\pi/2\\ \pi-x & \pi/2<x<\pi\end{cases}$. Hence deduce the sum of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. [AU A/M 2015]

9. (a). Find the half range sine series $f(x)=lx-x^2\ in\ (0,l)$ [AU Nov 2013]

(b). Find the half-range cosine series for $f(x)=(x-1)^2\ in\ 0<x<1$.

[A.U. MAY 2013, Nov 2014]

10. (a). Obtain fourier cosine series of $f(x)=x$, $0<x<4$ hence deduct that

$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+...+\infty$ [AU NOV 2014] (b). Obtain the Fourier series for the function $f(x)=\begin{cases} x & for\ 0 \le x \le 1 \\ 2-x & for\ 1 \le x \le 2 \end{cases}$

[A.U MAY 2011, 2013]

11. (a). Obtain the Fourier series for the function $f(x)=\begin{cases} l-x & for\ 0 \le x \le l \\ 0 & for\ l \le x \le 2l \end{cases}$ [A.U. N/D 2017]

(b). Obtain the sine series for the function $f(x)=\begin{cases} x & for\ 0 \le x \le l/2 \\ l-x & for\ l/2 \le x \le l \end{cases}$

[A.U. May 2011, 2013]

12. (a). Find the Fourier series expansion of f(x)=1-x^2 in the interval (-1,1)

[A.U DEC 2010]

(b). Find the half range sine series of $f(x)=x\cos\pi x \quad in \quad (0,1)$.

[A.U. M/J 2016]

13. (a). Find the complex form of the Fourier series of $f(x)= e^{-x}\ in\ -1<x\le 1$

[A.U. N/D 2009, A/M 2015]

(b). Find the complex form of Fourier series for the function $f(x)=e^{ax}$ in the interval $(-\pi,\pi)$ Where a is a real constant. Hence, deduce that $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{\pi}{a\sinh a\pi}$.

[A.U.N/D 2015]

14. (a). Find the complex form of the Fourier series $f(x)=\cos ax \quad in\ -\pi<x<\pi$.

[A.U. MAY 2013]

(b). Compute upto first harmonics of the fourier series of f(x) given by the following table:

x	0	T/6	T/3	T/2	2T/3	5T/6	T
f(x)	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

[A.U. NOV 2009]

15. (a). Determine the first two harmonics of Fourier series for the following data:

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
$f(x)$:	1.98	1.30	1.05	1.30	-0.88	-0.25

[A.U.N/D 2015]

(b). Find the Fourier series as far as the second harmonic to represent the function f(x) with period given in the following table.

X	0	1	2	3	4	5
f(x)	9	18	24	28	26	20

[A.U DEC 2010, 2012]

16.(a). Find the Fourier series up to second harmonic for y = f(x) from the following values

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1

[A.U MAY 2015, 2014,2017,2018]

(b). Find the Fourier cosine series up to third harmonic to represent the function given by the following data:

X	0	1	2	3	4	5
f(x)	4	8	15	7	6	2

[A.U. M/J 2016]

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II Year B.E (Civil, EEE & Mech.)

DEPARTMENT OF MATHEMATICS

SUBJECT NAME : MA8353 - TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS

Unit III (APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS)

Syllabus:

Classification of PDE – Method of separation of variable – Solutions of one dimensional wave equation – One dimensional heat conduction –Steady state solution of two dimensional equation of heat conduction(excluding insulated edges).

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	26	75	25	71

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PRINCIPAL

PART-A

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

1. What are the possible solutions of one-dimensional wave equations?

[AU –M/J 2006, Nov/Dec 2009/MAY-2014]

Solution:

$$y(x,t) = (c_1 e^{px} + c_2 e^{-px})((c_3 e^{pat} + c_4 e^{-pat})$$

$$y(x,t) = (c_5 \cos px + c_6 \cos px)((c_7 \cos pat + c_8 \sin pat$$

$$y(x,t) = (c_9 x + c_{10})((c_{11} t + c_{12})$$

2. In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ **what does** c^2**?** **[AU Nov/Dec 2011, June 2013]**

Solution: $c^2 = \frac{Tension}{Mass}$

3. What is the basic difference between the solution of one dimensional wave equation and one dimensional heat equation? **[AU-N/D 2017]**

Solution:

Solution of one dimensional wave equation is of periodic in nature. But solution of the one dimensional heat equation is not of periodic in nature.

4. Classify the Partial differential equation

$$(1 - x^2) z_{xx} - 2xyz_{xy} + (1 - y^2) z_{yy} + xz_x + 3x^2 yz_y - 2z = 0$$

Solution: **[AU – Nov-2014, May - 2015]**

Here $A = (1 - x^2) \quad B = -2xy \quad C = (1 - y^2)$

$$\begin{aligned} B^2 - 4AC &= (-2xy)^2 - 4[(1 - x^2)(1 - y^2)] \\ &= 4x^2 y^2 - 4[1 - x^2 - y^2 + x^2 y^2] \\ &= 4x^2 y^2 - 4 + 4x^2 + 4y^2 - 4x^2 y^2 \\ &= 4x^2 + 4y^2 - 4 \end{aligned}$$

$$\left.\begin{aligned} x > 1, y > 1 \\ x < -1, y < -1 \end{aligned}\right\} = B^2 - 4AC > 0 (+ve) \therefore \ Hyperbolic$$

$$When \quad x = 0, y = 0, \ B^2 - 4AC < 0 \ (-ve) \therefore \ Elliptic$$

5. A rod 30 cm long has its ends A and B kept at 20℃ and 80℃ respectively until steady state conditions prevail. Find the steady state temperature in the rod.

Solution: **[AU-May 2009, Apr 2008, May-2015]**

Let l=30 cm.

When the steady state condition prevail the heat flow equation is $\frac{\partial^2 u}{\partial x^2} = 0$

$u(x) = ax + b$.......... ....(1)

When the steady state condition exists the boundary conditions are

$$u(0) = 20; \; u(l) = 80 ....(2)$$

Applying (2) in (1), we get

$$\text{A=20}, \quad u(0) = b = 20.$$

$$a = \frac{60}{l}$$

$$u(x) = \frac{60}{l}x + 20, l = 30\,cm$$

6. State the laws assumed to derive the one dimensional heat equation. (OR) State the assumption in deriving the one dimensions heat flow equation (Unsteady State).

[AU- MAY /2014]

Solution:

1 .Heat flows from a higher temperature to lower temperature.

2. The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change.

3. The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the curve.

7. Given 3 possible solutions of the equation $\frac{\partial^2 y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2}$ (Or) Write down the various possible solutions of one dimensional heat flow equation.

[AU. Nov – 2014, M/J 2016,N/D 2016,A/M 2017, A/M 18]

$(i) \quad u(x,t) = (c_1 x + c_2)$

$(i) \quad u(x,t) = e^{\alpha^2 p^2 t}(c_3 e^{px} + c_4 e^{-px})$

$(i) \quad u(x,t) = e^{-\alpha^2 p^2 t}(c_5 \cos px + c_6 \sin px)$

8.Explain the term “steady state”. **[A.U.NOV 2013]**

Solution:

When the heat flow is independent of time't', it is called steady state. in steady state the heat flows only with respect to the distance 'x'.

9. write down the p.d.e equation that represents steady state heat flow two dimensional and name the variables involved. **[A.U.M/J 2012]**

Solution: $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

10.A rod 40 cm long with insulated sides with insulated sides has its ends A and B kep at $20^\circ c$ and $60^\circ c$ .Find the steady state temperature at a location 15 cm from A. **[A.U. A/M 2011]**

Solution.

$$u(x)=\left(\frac{b-a}{l}\right)x+a\,,0<x<40$$

$$u(x)=\left(\frac{60-20}{40}\right)x+20\,,0<x<40$$

$$u(15)=15+20=35$$

11. write down the three possible solutions of Laplace equations in two-dimensions.

Solution: **[A.U. N/D 2010, 2011, 17]**

$$y(x,y)=(c_1e^{px}+c_2e^{-px})((c_3\cos py+c_4\sin py)$$

$$y(x,t)=(c_5\cos px+c_6\sin px)((c_7e^{py}+c_8e^{-py})$$

$$y(x,t)=(c_9x+c_{10})((c_{11}y+c_{12})$$

12. Write down the boundary conditions for the following boundary value problem "If a string of length $'l'$ initially at rest in its equilibrium position and each of its point is given the velocity $\left(\frac{\partial y}{\partial t}\right)_t=0=v_0\sin^3\left(\frac{\pi x}{l}\right) in\ 0<x<l$

Determine the displacement function $y(x,t)$? **[A. U. A/M 2010]**

Solution:

The boundary conditions are

(i) $y(0,t)=0,t>0$

(ii) $y(l,t)=0,t>0$

(iii) $y(x,0)=0,0<x<t$

(iv) $\left(\dfrac{\partial y(x,0)}{\partial t}\right) = 0 = v_0 \sin^3\left(\dfrac{\pi x}{l}\right) in\ 0 < x < l$

13. Classify the Partial differential equation $4\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ **[A.U. N/D 2009]**

Solution:

$$4\frac{\partial^2 u}{\partial^2 x^2} = \frac{\partial u}{\partial t}$$

A=4, B=0, C=0

$$B^2 - 4AC = 0 - 4(0)(0) = 0$$

14. Classify the Partial differential equation $U_{xy} = U_x U_y + xy$

Solution: **[AU –A/M 2018]**

Here $A = 0 \quad B = 1 \quad C = 0$

$B^2 - 4AC = (1)^2 \succ 0$

Which is hyperbolic equation.

15. Classify the Partial differential equation. **[AU – A/M-2008]**

$$3U_{xx} + 4U_{xy} - 2U_x + 3U_y = 0$$

Solution:

$A = 3 \quad B = 4 \quad C = 0$

$B^2 - 4AC = (4)^2 - 4(3)(0) = 16 > 0$

Which is heperbolic equation.

16. A rod 50 cm long has its ends A and B kept at 20 and 70 degree respectively until steady state conditions prevail. Find the steady state temperature in the rod.

Solution: **[AU-,N/D 2008]**

When the steady state condition prevail the heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} = 0$

$u(x) = ax + b$.......... ....(1)

When the steady state condition exists the boundary conditions are

$u(0) = 20;\ u(l) = 70$.......... ......(2)

Put x=0 in (1) $u(x) = 0x + b \Rightarrow b = 20$

$$u(l) = al + b = 70$$

Put x= $x = l$ in (1) $= al + 20 = 70 \Rightarrow a = \frac{50}{l}$

Applying (2) in (1), we get

$$a = \frac{50}{l}, \quad b = 20. \qquad l = 50$$

$$u(x) = x + 20$$

17. A rod 10 cm long has its ends A and B kept at 20 and 70 degree respectively until steady state conditions prevail. Find the steady state temperature in the rod.

Solution: **[AU-,M/J 2008]**

When the steady state condition prevail the heat flow equation is $\frac{\partial^2 u}{\partial x^2} = 0$

$$u(x) = ax + b.......... ....(1)$$

When the steady state condition exists the boundary conditions are

$$u(0) = 20; \ u(l) = 70.......... ......(2)$$

Put x=0 in (1) $u(x) = 0x + b \Rightarrow b = 20$

$$u(l) = al + b = 70$$

Put x= $x = l$ in (1) $= al + 20 = 70 \Rightarrow a = \frac{50}{l}$

Applying (2) in (1), we get

$$a = \frac{50}{l}, \quad b = 20. \qquad l = 10$$

$$u(x) = 5x + 20$$

18. In steady state conditions derive the solution of one dimensional heat flow equation.

[AU-Nov 2005, M/J 2006]

Solution:

When the steady state conditions exists the heat flow equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad ....(1)$$

In a steady state condition, the temperature u depends only on x and not time t

Hence $\frac{\partial u}{\partial t} = 0$

(1) Reduces $\frac{\partial^2 u}{\partial x^2} = 0$ ……………………………………………………………(2)

The general solution is $u(x) = ax + b$

19. State one dimensional heat equation with the initial and final boundary conditions.
Solution: **[AU-N/D 2006]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

$u(x,t)$ is the temp, time t at a point distance x from the left end of the rod.

The following boundary &initial conditions

(i) $u(0,t) = k_1{}^{\bullet}c \quad \forall\, t \ge 0$

(ii) $u(l,t) = k_2{}^{\bullet}c \quad \forall\, t \ge 0, \forall\, t \ge 0$

(iv) $u(x,0) = f(x) \quad 0 \le x \le l$

20. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subject to initial displacement f(x) and initial velocity g(x).

Solution: **[AU-N/D 2006,2007]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem , we get the following boundary &initial conditions**.**

(i) $y(0,t) = 0, \forall\, t \ge 0$

(ii) $y(l,t) = 0, \forall\, t \ge 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = g(x) \ \forall\ 0 < x < l$

(iv) $y(x,0) = f(x)$

21. Classify the Partial differential equation

$$y^2 U_{xx} - 2xyU_{xy} + x^2 U_{yy} + 2U_x - 3U_y = 0$$

Solution: **[AU – Nov-2009]**

Here $A = y^2 \quad B = -2xy \quad C = x^2$

$B^2 - 4AC = (-2xy)^2 - 4(y^2)(x^2) = 4x^2y^2 - 4x^2y = 0$

Hence the function is parabolic

22. An insulated rod of length l=60 cm has its ends at A and B maintained at 30°C and 40°C respectively. Find the steady state solution

[AU – Nov-2012]

Solution:

The heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

When steady state condition exist the heat flow equations becomes $\dfrac{\partial^2 u}{\partial x^2} = 0$

When the steady state condition prevail the heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} = 0$

$$u(x) = ax + b.......... \quad .... \quad .... \quad ....(1)$$

When the steady state condition exists the boundary conditions are

$$u(0) = 30; \quad u(l) = 40..........\ ......(2)$$

Put x=0 in (1) $u(0) = 0x + b \Rightarrow b = 30$

Put x=l in (1) $u(l) = al + b = 40$

$$al + 30 = 40 \Rightarrow a = \frac{40}{l}$$

$$a = \frac{40}{l}, \quad b = 30. \ l = 60$$

$$u(x) = \frac{40}{l}x + 30$$

23. A rectangular plate is bounded by the linear line x=0,y=0,x=l and y=l .Its surface is insulated. The edge coinciding with x-axis is kept at $100^\bullet c$. The edge coinciding with y-axis is kept at $50^\bullet c$ .The other two edges are kept at $0^\bullet c$. Write down the boundary conditions that are needed for solving two dimensional heat flow equation.

[AU Nov/Dec 2012, 2011]

Solution:

The two dimensional wave equation is $\dfrac{\partial^2 y}{\partial x^2} + \dfrac{\partial^2 y}{\partial y^2} = 0$

From the given problem, we get the following boundary &initial conditions**.**

(i) $u(x,0) = 100^\bullet C, \forall \quad 0 < x < l$

(ii) $u(0, y) = 50^\bullet C, \forall \quad 0 < x < l$

(iii) $u(x,l) = 0^\bullet C \quad \forall \quad 0 < x < l$

(iv) $u(l, y) = 0^\bullet C \quad \forall \quad 0 < x < l$

24. Write down the two dimensional heat equation both in transient and Steady state.

Solution: **[AU May/Jun 2012]**

$$\frac{\partial u}{\partial t} = \frac{K}{\rho C}\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right) = 0$$ **Transient state**

$$\left(\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right) = 0$$ **Steady state**

25. Write down the boundary conditions and initial conditions when a taut string of length 2l is fastened on both ends. The midpoint of the string is taken to a height b and released from the rest in that position.

Solution. **[AU-N/D 2015]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem , we get the following boundary &initial conditions**.**

(i) $y(0,t) = 0, \ \forall \ t \geq 0$

(ii) $y(2l,t) = 0, \ \forall \ t \geq 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \ \forall \ 0 < x < 2l$

(iv) $y(x,0) = \begin{cases} \frac{bx}{l}, & 0 < x < l \\ \frac{b(2l - x)}{l}, & l < x < 2l \end{cases}$

26. Explain the initial and Boundary value problems. **[AU-Apr 2009]**
Solution:

In Ordinary differential equations, first we get the general solution which contains the arbitrary constants and then we determine these constants from the given initial values. These types of problem are called initial value problems.

In many physical problems, we always seek a solution of the differential equations, whether it ordinary or partial, which satisfies some specified equations called boundary conditions. Any differential equations together with these boundary conditions are called boundary value problems.

27. State the assumptions made in the derivation of one dimensional wave equation.
Solution: **[AU N/D 2016]**

(i) The mass of the string per unit length is constant.

(ii) The string is perfectly elastic and does not offer any resistance to bending.

(iii) The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected.

28. State Fourier law of conduction. **[AU-Apr 2010]**

Solution:

The rate at which heat flows across an area A at a distance x from one end of a bar is given by

$$Q = iKA\left(\frac{\partial u}{\partial x}\right)x$$

K is Thermal conductivity and $\left(\frac{\partial u}{\partial x}\right)x$ means the temperature gradient at x.

29. Distinguish between steady and unsteady states condition in one dimensional heat flow equation.

Solution:

	STEADY STATE	**UNSTEADY STATE**
1.	Temperature depends only on distance	Temperature depends on distance 'x' and time 't'
2.	Equation is $\frac{\partial^2 u}{\partial x^2} = 0$	Equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

30.Write all possible solutions of two dimensional heat equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0$

[AU-N/D 2015]

$$y(x,y) = (c_1 e^{px} + c_2 e^{-px})((c_3 \cos py + c_4 \sin py)$$

$$y(x,t) = (c_5 \cos px + c_6 \sin px)((c_7 e^{py} + c_8 e^{-py})$$

$$y(x,t) = (c_9 x + c_{10})((c_{11} y + c_{12})$$

31. By the method of separation of variables solve $3x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$

Solution: **[AU-N/D 2015]**

Given $3x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$ ………………………………(1)

Let Z = X(x) Y(y) be the solution of (1).

$$3xYX' - 2yXY' = 0$$

$$3x\frac{X'}{X} = 2y\frac{Y'}{Y} = K(say)$$

$$3x\frac{X'}{X} = k \Rightarrow \frac{X'}{X} = \frac{k}{3x}$$

$$\frac{1}{X}\frac{dX}{dx} = \frac{k}{3x}$$

$$\int \frac{1}{X}dX = \int \frac{k}{3x}dx$$

$$\log X = \frac{k}{3}\log x + \log c_1$$

$$2y\frac{Y^{'}}{Y} = K$$

Similarly the above steps

$$\log Y = \frac{k}{3}\log y + \log c_2$$

32 . Classify the Partial differential equation $\frac{\partial^2 u}{\partial xx} + \frac{\partial^2 u}{\partial xy} = f(x, y)$

[A.U. N/D 2009,2016]

Solution:

$$\frac{\partial^2 u}{\partial xx} + \frac{\partial^2 u}{\partial xy} = f(x, y)$$

A=1, B=1, C=0

$$B^2 - 4AC = 1^2 - 4(1)(0) = 1(+ve)$$

Which is a hyperbolic

33. By the method of separation of variables solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where u(x,0) = $6e^{-3x}$

Solution: **[AU-M/J 2017]**

Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

$$X^{'}T = 2XT^{'} + XT = 0$$

$$\frac{X^{'}}{X} = \frac{2T^{'} + T}{T} = K(say)$$

$$\frac{X^{'}}{X} = k \Rightarrow X^{'} = KX$$

$$\frac{1}{X}\frac{dX}{dx} = kdx$$

$$\int \frac{1}{X}dX = \int kdx$$

$$\log = kx + \text{loga}$$

PART-B

1. ZERO INITIAL VELOCITY (METHOD-1)

Key Words (I) Displacement of any point of the string (II) Initially Displacement

1. Tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$ apart. Motion is Started by displacing the string in to the from $y=k(lx-x^2)$ from which it is released at time t=0. Find the displacement of any point of the string at a distance of x from one end at time t. **[AU - May -2013,N/D2013, MAY-2015, N/D 17, A/M 18]**

Solution:

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary &initial conditions.

(i) $y(0,t)=0,\ \forall\ t\geq 0$

(ii) $y(l,t)=0,\ \forall\ t\geq 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)}=0,\ \forall\ 0<x<l$

(iv) $y(x,0)=\ k(lx-x^2)\quad 0\leq x\leq l$

The correct solution on the boundary condition is

$$y(x,t)=(c_1\cos px+c_2\sin px)\,(c_3\cos pat+c_4\sin pat)........(1)$$

Applying condition (i) in (1), we get

$$y(0,t)=c_1\,(c_3\cos pat+c_4\sin pat)=0$$

$$c_1=0\ ,\ (c_3\cos pat+c_4\sin pat)\neq 0$$

Substituting $c_1=0\ \ in\ eqn\,(1)$

$$y(x,t)=c_2\sin px\,(c_3\cos pat+c_4\sin pat)\quad ....(2)$$

Applying condition (ii) in (2), we get

$$y(l,t)=c_2\sin pl\,(c_3\cos pat+c_4\sin pat)=0$$

Here, $(c_3\cos pat+c_4\sin pat)\neq 0$

Therefore, either $c_2=0\quad (or)\ \sin pl=0$

Suppose, we take $c_2=0$ and already we have $c_1=0$ then we get a trivial Solution.

we consider $c_2\neq 0\ \ and\ \ \sin pl=0$

$$\sin pl = 0 \ , pl = \sin n\pi \ , pl = n\pi \qquad \Rightarrow p = \frac{n\pi}{l}$$

Substituting $p = \frac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}\right) \quad \text{..............................(I)}$$

Differentiating (3) partially w. r. t 't'.

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(c_3 \frac{n\pi a}{l}\left(-\sin\frac{n\pi at}{l}\right) + c_4 \frac{n\pi a}{l}\cos\frac{n\pi at}{l}\right) \quad \text{..............(3)}$$

Applying condition (iii in (3) , we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi a}{l}\right) = 0$$

Here $c_2 \neq 0$ *already explained* ,

$\sin\frac{n\pi x}{l} \neq 0$ *it is defined forall x* , $\frac{n\pi x}{l} \neq 0$ *all are constants*

there fore $c_4 = 0$

Substituting $c_4 = 0$ **in (I) we get**

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{, Where } c_2 c_3 = c_n$$

The most general solution (4) is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{...(4)}$$

Applying condition (iv) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = k(lx - x^2) \quad \text{..(5)}$$

To find expand C_n: expand $k(lx - x^2)$ **in a half range Fourier sine series in the interval** $0 \le x \le l$

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{..(6)}$$

$$\text{where } b_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}dx$$

From equations (5) and (6) , we get $b_n = c_n$

$$c_n = \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}dx$$

$$c_n = \frac{2k}{l}\left[(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{l} + (-2)\left(\frac{l^3}{n^3\pi^3}\right)\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2k}{l}\left[\left\{0 + 0 - \frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right\}\right]$$

$$= \frac{2k}{l}\left(\frac{2l^3}{n^3\pi^3}\right)[1 - \cos n\pi] = \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$c_n = \begin{cases} 0 & when\ n\ is\quad even \\ \dfrac{8kl^2}{n^3\pi^3} & when\ n\ is\quad odd \end{cases}$$

Substituting the value of c_n in equation (4) $y(x,t) = \sum_{n=odd}^{\infty} \frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$

Result: $y(x,t) = \sum_{n=even}^{\infty} \frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$

2. A Tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At the time t=0 , the string is given by a shape defined by $f(x) = kx^2(l - x)$, where k is constant , and then released from rest . Find the displacement of any point 'x' of the string at any time t>0**.**

Solution: **[AU / MAY -2010,2008]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary &initial conditions.

(i) $y(0,t) = 0,\ \forall\ t \geq 0$

(ii) $y(l,t) = 0,\ \forall\ t \geq 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0,\ \forall\ 0 < x < l$

(iv) $y(x,0) = kx^2(l - x) \quad 0 \leq x \leq l$

The correct solution on the boundary condition is

$$y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos pat + c_4\sin pat)........(1)$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1\,(c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 = 0\ ,\ (c_3 \cos pat + c_4 \sin pat) \neq 0$$

Substituting $c_1 = 0 \ \ in\ eqn\,(1)$

$$y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0$ *and* $\sin pl = 0$

$$\sin pl = 0\ ,\ pl = \sin n\pi\ ,\ pl = n\pi \quad \Rightarrow p = \frac{n\pi}{l}$$

Substituting $p = \dfrac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

Differentiating (I) partially w. r. t 't'.

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left(c_3 \frac{n\pi a}{l}\left(-\sin \frac{n\pi at}{l}\right) + c_4 \frac{n\pi a}{l} \cos \frac{n\pi at}{l}\right) \quad \ldots\ldots\ldots\ldots\text{(3)}$$

Applying condition (iii in (3) , we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left(c_4 \frac{n\pi a}{l}\right) = 0$$

Here $c_2 \neq 0$ *already explained* ,

$$\sin \frac{n\pi x}{l} \neq 0 \ \ it\ is\ defined\ \ forall\ x \quad , \frac{n\pi x}{l} \neq 0\ \ all\ are\ constants$$

there fore $c_4 = 0$

Substituting $c_4 = 0$ **in (I) we get**

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

$$= c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l} \quad \text{, Where } c_2 c_3 = c_n$$

The most general solution is

$$y(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \text{ ....(4)}$$

Applying condition (iv) in (4) we get

$$y(x,0)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{l}=kx^2(l-x) \text{ ....(5)}$$

To find expand $kx^2(l-x)$ **in a half range Fourier sine series in the interval** $0\le x\le l$

$$kx^2(l-x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \text{ ....(6)}$$

$$where\ \ b_n=\frac{2}{l}\int_0^l kx^2(l-x)\sin\frac{n\pi x}{l}dx$$

From equations (5) and (6) ,we get $b_n=c_n$

$$c_n=\frac{2}{l}\int_0^l kx^2(l-x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2k}{l}\left[\begin{matrix}(lx^2-x^3)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)-(2lx-3x^2)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)+(2l-6x)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\\+6\left(\frac{l^4}{n^4\pi^4}\sin\frac{n\pi x}{l}\right)\end{matrix}\right]_0^l$$

$$=\frac{2k}{l}\left[-4l\left(\frac{l^3}{n^3\pi^3}\cos n\pi\right)-\frac{l^3}{n^3\pi^3}(2l)\right]$$

$$=\frac{2k}{l}\times\frac{l^3}{n^3\pi^3}\left[-4l\cos n\pi-2l\right]$$

$$c_n=\frac{-4kl^3}{n^3\pi^3}\left[1+2(-1)^n\right] \text{ ....(7)}$$

Substituting (7) in (4) , We get

$$y(x,t)=\sum_{n=1}^{\infty}\frac{-4kl^3}{n^3\pi^3}\left[1+2(-1)^n\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

RESULT: $y(x,t)=\sum_{n=1}^{\infty}\frac{-4kl^3}{n^3\pi^3}\left[1+2(-1)^n\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$

3. A Tightly stretched string with fixed end point $x=0$ and $x=l$ is initially in a position given by $y(x,0)=y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position.

Find the displacement y at any distance x from one end at any time t. **[AU / DEC -2012]**

Solution:

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary &initial conditions.

(i) $y(0,t) = 0, \forall\ t \geq 0$

(ii) $y(l,t) = 0, \forall\ t \geq 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \forall\ 0 < x < l$

(iv) $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}. \quad 0 \leq x \leq l$

The correct solution on the boundary condition is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) ....(1)$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 = 0 \text{ , } (c_3 \cos pat + c_4 \sin pat) \neq 0$$

Substituting $c_1 = 0$ $in\ eqn\ (1)$

$$y(x,t) = c_2 \sin px (c_3 \cos pat + c_4 \sin pat) \text{ ....(2)}$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl (c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0$ and $\sin pl = 0$

$\sin pl = 0$, $pl = \sin n\pi$, $pl = n\pi$ $\Rightarrow p = \frac{n\pi}{l}$

Substituting $p = \frac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \text{ ....(I)}$$

Differentiating (I) partially w. r. t 't'.

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\,(c_3 \frac{n\pi a}{l}(-\sin\frac{n\pi at}{l}) + c_4 \frac{n\pi a}{l}\cos\frac{n\pi at}{l}) \quad \text{.............(3)}$$

Applying condition (iii in (3) , we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\,(c_4 \frac{n\pi a}{l}) = 0$$

Here $c_2 \neq 0$ *already explained* ,

$\sin\frac{n\pi x}{l} \neq 0$ *it is defined forall x* , $\frac{n\pi x}{l} \neq 0$ *all are constants*

there fore $c_4 = 0$

Substituting $c_4 = 0$ **in (I) we get**

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{, Where } c_2 c_3 = c_n$$

The most general solution (4) is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{...(4)}$$

Applying condition (iv) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l} \quad \text{.......................................(5)}$$

We know that $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$$\sin^3\frac{\pi x}{l} = \frac{1}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right) \quad \text{..............................(6)}$$

From (5) and (6) we get

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

$$c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{2\pi x}{l} + c_3 \sin\frac{3\pi x}{l} + \ldots.. = \frac{y_0}{4}\left(3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right)$$

Equating like coefficients on either side, we get $c_1 = \frac{3y_0}{4}$,

$c_2 = 0\,, c_3 = \frac{-y_0}{4}\,, c_4 = 0\,, c_5 = 0\,, c_6 = 0$

$$y(x,t) = c_1 \sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + c_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l} + c_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + c_4 \sin\frac{4\pi x}{l}\cos\frac{4\pi at}{l} .....(5)$$

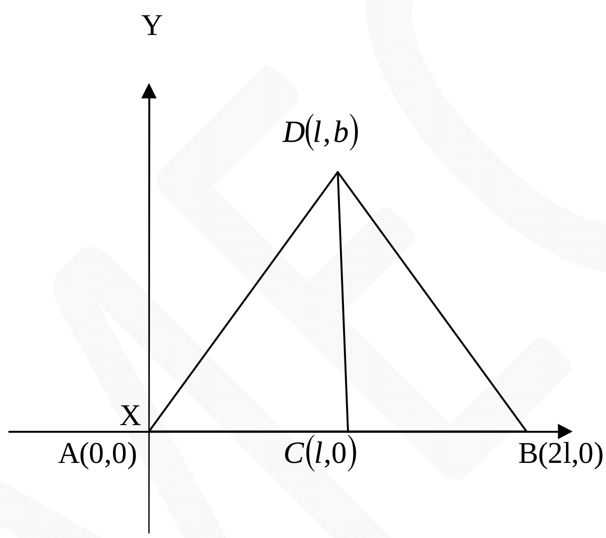
Substituting the above values of $c_1 = \frac{3y_0}{4}$, $c_2 = 0, c_3 = \frac{-y_0}{4}, c_4 = 0, c_5 = 0, c_6 = 0$ $in(5)$

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

4. A string of length 2l is fastend at both ends. The mid point of the string is taken to a height b and then released from rest in that position. Show that the displacement is

$$y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-2)^2}\sin\left(\frac{(2n-1)\pi x}{2l}\right)\cos\left(\frac{(2n-1)\pi at}{2l}\right)$$ **[AU A /M -2017]**

Solution:



Equation of AD is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{b - 0} = \frac{x - 0}{l - 0}$$

$$y = \frac{b}{l}x$$

Equation of DB is

$$\frac{y - b}{0 - b} = \left(\frac{x - l}{l}\right)(-b) \qquad y = b + \frac{bl}{l} - \frac{bx}{l} \qquad = 2b - \frac{bx}{l} \quad y = b\left[\frac{2l - x}{l}\right] ,$$

$$y(x,t)=\begin{cases}\dfrac{bx}{l}, & 0<x<l\\ \dfrac{b(2l-x)}{l}, & l<x<2l\end{cases}$$

The one dimensional wave equation is $\dfrac{\partial^2 y}{\partial t^2}=a^2\dfrac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary &initial conditions.

(i) $y(0,t)=0,\ \forall\ t\geq 0$

(ii) $y(2l,t)=0,\ \forall\ t\geq 0$

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{(x,0)}=0,\ \forall\ 0<x<2l$

(iv) $y(x,0)=\begin{cases}\dfrac{bx}{l}, & 0<x<l\\ \dfrac{b(2l-x)}{l}, & l<x<2l\end{cases}$

The correct solution on the boundary condition is

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat).........(1)$$

Applying condition (i) in (1), we get

$$y(0,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$

$$c_1=0\ ,\ (c_3\cos pat+c_4\sin pat)\neq 0$$

Substituting $c_1=0\ \ in\ eqn\,(1)$

$$y(x,t)=c_2\sin px\,(c_3\cos pat+c_4\sin pat)\ ....(2)$$

Applying condition (ii) in (2), we get

$$y(2l,t)=c_2\sin 2pl\,(c_3\cos pat+c_4\sin pat)=0$$

Here, $(c_3\cos pat+c_4\sin pat)\neq 0$

Therefore, either $c_2=0$ (or) $\sin 2pl=0$

Suppose, we take $c_2=0$ and already we have $c_1=0$ then we get a trivial Solution.

we consider $c_2\neq 0\ \ and\ \ \sin 2pl=0$

$$\sin 2pl=0\ ,2pl=\sin n\pi\ ,2pl=n\pi\qquad\Rightarrow p=\frac{n\pi}{2l}$$

Substituting $p=\frac{n\pi}{2l}$ **in (2) we get**

$$y(x,t)=c_2\sin\frac{n\pi x}{2l}\left(c_3\cos\frac{n\pi at}{2l}+c_4\sin\frac{n\pi at}{2l}\right)\quad\text{.............................(I)}$$

Differentiating (I) partially w. r. t 't'.

$$\frac{\partial y(x,t)}{\partial t}=c_2\sin\frac{n\pi x}{2l}\left(c_3\frac{n\pi a}{2l}\left(-\sin\frac{n\pi at}{2l}\right)+c_4\frac{n\pi a}{2l}\cos\frac{n\pi at}{2l}\right)\quad\text{.............(3)}$$

Applying condition (iii in (3) , we get

$$\frac{\partial y(x,0)}{\partial t}=c_2\sin\frac{n\pi x}{2l}\left(c_4\frac{n\pi a}{2l}\right)=0$$

Here $c_2\neq 0$ *already* explained ,

$\sin\frac{n\pi x}{2l}\neq 0$ *it is defined forall* x , $\frac{n\pi a}{2l}\neq 0$ *all are constants*

there fore $c_4=0$

Substituting $c_4=0$ **in (I) we get**

$$y(x,t)=c_2c_3\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}$$

$$=c_n\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}$$, Where $c_2c_3=c_n$

The most general solution (4) is

$$y(x,t)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}\quad\text{.......................................(4)}$$

Applying condition (iv) in (4) we get

$$y(x,0)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{2l}\quad\text{..(5)}$$

To find expand f(x) in a half range Fourier sine series in the interval $0\leq x\leq 2l$

$$\begin{cases}\frac{bx}{l}, & 0<x<l\\ \frac{b(2l-x)}{l}, & l<x<2l\end{cases}=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{2l}\quad\text{...(6)}$$

From equations (5) and (6) ,we get $b_n=c_n$

$$c_n=\frac{1}{l}\int_0^{2l}f(x)\sin\frac{n\pi x}{2l}\,dx$$

$$= \frac{1}{l}\left[\int_0^l \frac{bx}{l}\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\frac{b(2l-x)}{l}\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{b}{l^2}\int_0^l x\sin\frac{n\pi x}{2l}\,dx + \frac{b}{l^2}\int_l^{2l}(2l-x)\sin\frac{n\pi x}{2l}\,dx$$

$$= \frac{b}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_0^l + \frac{b}{l^2}\left[(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[-x\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} + \left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi x}{2l}\right]_0^l + \frac{b}{l^2}\left[-(2l-x)\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l} - \left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi x}{2l}\right]_l^{2l}$$

$$= \frac{b}{l^2}\left(\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l}{n^2\pi^2}^2\sin\frac{n\pi}{2}\right) - (-0+0) + \frac{b}{l^2}\left[(0-0) - \left(\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} - \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{b}{l^2}\left[\left(\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] \quad \text{ie.., } c_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$c_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$c_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2} \text{ if n is odd.}$$

Substitute the value of c_n in equation (4) we get

$$\text{Re } sult \;: y(x,t) = \sum_{n=odd}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}$$

5. A taut string of length L has its ends $x=0$ and $x=l$ fixed . The point where $x=\frac{l}{3}$ is drawn aside a small distance h , the displacement $y(x,t)$ satisfies $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$. Determine $y(x,t)$ at any time t. [AU / MAY -2010]

solution:

First we find the equation of the string in its initial position.

The equation of the line (or string) OB

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{h-0}=\frac{x-0}{\frac{l}{3}-0}$$

$$y=\frac{3xh}{l},\quad 0<x<\tfrac{l}{3.}$$

The equation of the line (or string) BA is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

$$\frac{y-h}{-h}=\frac{x-\frac{l}{3}}{\frac{2}{3}l}$$

$$y=\frac{3h}{2l}(l-x).,\qquad \frac{l}{3}<x<l$$

$$f(x)=\begin{cases}\dfrac{3xh}{l} & 0<x<\tfrac{l}{3}\\[2ex] \dfrac{3h}{2l}(l-x) & ,\dfrac{l}{3}<x<l\end{cases}$$

The one dimensional wave equation is $\dfrac{\partial^2 y}{\partial t^2}=a^2\dfrac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary &initial conditions.

(i) $y(0,t)=0,\ \forall\ t\geq 0$

(ii) $y(l,t)=0,\ \forall\ t\geq 0$

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{(x,0)}=0,\ \forall\ 0<x<l$

(iv) $y(x,0)=\begin{cases}\dfrac{3xh}{l} & 0<x<\tfrac{l}{3}\\[2ex] \dfrac{3h}{2l}(l-x) & ,\dfrac{l}{3}<x<l\end{cases}$

The correct solution on the boundary condition is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos pat + c_4 \sin pat)\text{...........}(1)$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1\,(c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 = 0\ ,\ (c_3 \cos pat + c_4 \sin pat) \neq 0$$

Substituting $c_1 = 0$ $in\ eqn\,(1)$

$$y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \quad \text{.....................}(2)$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0$ and $\sin pl = 0$

$$\sin pl = 0\ ,\ pl = \sin n\pi\ ,\ pl = n\pi \quad \Rightarrow p = \frac{n\pi}{l}$$

Substituting $p = \dfrac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\,(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}) \quad \textbf{.............................(I)}$$

Differentiating (I) partially w. r. t 't'.

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\,(c_3\frac{n\pi a}{l}(-\sin\frac{n\pi at}{l}) + c_4\frac{n\pi a}{l}\cos\frac{n\pi at}{l}) \quad \textbf{.............(3)}$$

Applying condition (iii) in (3) , we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\,(c_4\frac{n\pi a}{l}) = 0$$

Here $c_2 \neq 0$ $already\ \text{explained}$,

$$\sin\frac{n\pi x}{l} \neq 0\ \ it\ is\ defined\ \ forall\ \ x \quad ,\frac{n\pi x}{l} \neq 0\ \ all\ are\ cons\tan ts$$

there fore $c_4 = 0$

Substituting $c_4 = 0$ **in (I) we get**

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{, Where } c_2 c_3 = c_n$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \text{ ....(4)}$$

Applying condition (iv) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \text{ ....(5)}$$

To find expand f(x) in a half range Fourier sine series in the interval $0 \le x \le l$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \begin{cases} \dfrac{3xh}{l} & 0 < x < \frac{l}{3} \\ \dfrac{3h}{2l}(l-x) & , \dfrac{l}{3} < x < l \end{cases} \text{ ....(6)}$$

From equations (5) and (6) ,we get $b_n = c_n$

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{3}} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{3}} \frac{3hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{l} \frac{3h}{2l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{3}} \frac{3hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{3}}^{l} \frac{3h}{2l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6h}{l^2}\left[\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{l/3} + \frac{1}{2}\left[-(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/3}^{l}\right]$$

$$= \frac{6h}{l^2}\left[\begin{array}{l}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}-(0+0)\right] \\ +\frac{1}{2}\left[(-0-0)-(-(l-\frac{l}{3})\frac{l}{n\pi}\cos\frac{n\pi}{3}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3})\right]\end{array}\right]$$

$$= \frac{6h}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}+\frac{1}{2}\frac{2l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{2n^2\pi^2}\sin\frac{n\pi}{3}\right]$$

$$= \frac{6h}{l^2}\left[\frac{3}{2}\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right]$$

$$c_n = \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3} \quad \text{..(7)}$$

Equation (7) in (4)

$$y(x,t) = \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l.}$$

RESULT:

$$y(x,t) = \frac{9h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l.}$$

6. A String is stretched and fastened to points at a distance l apart the motion is started by displace the string in form $y = a\sin\frac{\pi x}{l}, .0 < x < l$ from which it is released at a time t = 0 find the displacement at any time t [AU / MAY -2014]

Solution:

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary &initial conditions.

(i) $y(0,t) = 0, \ \forall\ t \ge 0$

(ii) $y(l,t) = 0, \ \forall\ t \ge 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \ \forall\ 0 < x < l$

(iv) $y(x,0) = \ a\sin\frac{\pi x}{l}. \quad 0 \le x \le l$

The correct solution on the boundary condition is

$$y(x,t) = (c_1\cos px + c_2\sin px)\,(c_3\cos pat + c_4\sin pat)\text{.........(1)}$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1\,(c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 = 0\ ,\ (c_3 \cos pat + c_4 \sin pat) \neq 0$$

Substituting $c_1 = 0$ *in eqn* (1)

$$y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0$ *and* $\sin pl = 0$

$$\sin pl = 0\ ,\ pl = \sin n\pi\ ,\ pl = n\pi \quad \Rightarrow p = \frac{n\pi}{l}$$

Substituting $p = \frac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\,(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(I)$$

Differentiating (3) partially w. r. t 't'.

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\,(c_3 \frac{n\pi a}{l}(-\sin\frac{n\pi at}{l}) + c_4 \frac{n\pi a}{l}\cos\frac{n\pi at}{l}) \quad \ldots\ldots\ldots\ldots(3)$$

Applying condition (iii) in (I) , we get

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\,(c_4 \frac{n\pi a}{l}) = 0$$

Here $c_2 \neq 0$ *already explained* ,

$$\sin\frac{n\pi x}{l} \neq 0 \ \textit{it is defined forall } x \quad , \frac{n\pi x}{l} \neq 0 \ \textit{all are constants}$$

there fore $c_4 = 0$

Substituting $c_4 = 0$ **in (I) we get**

$$y(x,t) = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{, Where } c_2c_3 = c_n$$

The most general solution (4) is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{...(4)}$$

Applying condition (iv) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l} \quad \text{......................................(5)}$$

The most general solution is

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l} \quad \text{...........(6)}$$

$$c_1\sin\frac{\pi x}{l} + c_2\sin\frac{2\pi x}{l} + \ldots\ldots = a\sin\frac{\pi x}{l}$$

Equating the coefficients, we get

$c_1 = a$, $c_2 = 0$, $c_3 = 0$

$$y(x,t) = c_1\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + c_2\sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l} + c_3\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + c_4\sin\frac{4\pi x}{l}\cos\frac{4\pi at}{l}\ldots..(C)$$

Substituting $c_1 = a$, $c_2 = 0$, $c_3 = 0$ in (3) we get

$$y(x,t) = a\sin\frac{\pi x}{l}\cos\frac{\pi at}{l}$$

2. NON ZERO VELOCITY (METHOD-2)

KEY WORDS (i) Equilibrium Position (ii) Given velocity (iii) Initially at rest

7. A tightly stretched string with fixed end points x = 0 and x = 1 is initially at rest in its equilibrium Position. It is set vibrating string giving each point a velocity $\lambda x(1-x)$

Solution: **[AU / MAY -2013]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary conditions are

(i) $y(0,t) = 0, \forall t > 0$

(ii) $y(l,t) = 0, \forall t > 0$

(iii). $y(x,0) = 0, 0 < x < l$

(iv). $\frac{\partial y}{\partial t}(x,0) = \lambda x(l-x) = f(x), 0 < x < l.$

The correct solution on the boundary condition is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos pat + c_4 \sin pat) \text{..........(1)}$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1\,(c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 = 0\,,\ (c_3 \cos pat + c_4 \sin pat) \neq 0$$

Substituting $c_1 = 0$ $in\ eqn\,(1)$

$$y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \quad \text{.....................(2)}$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0$ and $\sin pl = 0$

$\sin pl = 0$, $pl = \sin n\pi$, $pl = n\pi$ $\Rightarrow p = \frac{n\pi}{l}$

Substituting $p = \frac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}\,(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}) \quad \textbf{..............................(3)}$$

Applying condition (iii) in equation (3) we get

$$y(x,0) = c_2 \sin \frac{n\pi x}{l} c_3 = 0$$

Here $\sin \frac{n\pi x}{l} \neq 0$ [$\because$ It is defined for all x]

$c_2 \neq 0$ [$\because$ $c_2 = 0$ we already explained]

Therefore $c_3 = 0$

Substitute $C_3 = 0$ in equation (3) we get

$$y(x,t) = c_2 c_4 \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}$$

$$= c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l}, \qquad where \quad c_2 c_4 = C_n$$

The most general solution we get

$$y(x,t)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad \text{.............................(I)}$$

Partially diff w. r. to 't' we get

$$\frac{\partial y}{\partial t}(x,t)=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) \text{----------------------------(4)}$$

Now we apply condition (iv) in (4) we get

$$\frac{\partial y}{\partial t}(x,0)=\sum_{n=1}^{\infty} C_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l}=\lambda x(l-x) \quad \text{..................................(5)}$$

Now to find c_n expand $\lambda x(l-x)$ in a half-range Fourier series , we get

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}=\lambda x(l-x) \quad \text{.......................................(6)}$$

From equations (5) and (6) we get

$$\sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l}=\sum_{n=1}^{\infty} b_n in\frac{n\pi x}{l}$$

Equating like co-efficients, we get

$$C_n=b_n\left(\frac{n\pi a}{l}\right) \quad \text{...(7)}$$

$$b_n=\frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2\lambda}{l}\left[(lx-x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]-(l-2x)\left[\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]+(-2)\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^l$$

$$=\frac{2\lambda}{l}\left[-(lx-x^2)\left(\frac{l}{n\pi}\right)\cos\frac{n\pi x}{l}+(l-2x)\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}-2\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{2\lambda}{l}\left[\left(0+0-2\left(\frac{l}{n\pi}\right)^3(-1)^n\right)-\left(0+0-2\left(\frac{l}{n\pi}\right)^3\right)\right]$$

$$=\frac{2\lambda}{l}\left[-2\left(\frac{l}{n\pi}\right)^3(-1)^n+2\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2\lambda}{l} 2\left(\frac{l}{n\pi}\right)^3 \left[1-(-1)^n\right] = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$c_n\left(\frac{n\pi a}{l}\right) = \frac{4\lambda l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$c_n = 0$ if n is even.

$c_n = \frac{8\lambda l^2}{n^3\pi^3}$ is n is odd……………………………………(8)

Substituting (8) in (7)

$$c_n = b_n \frac{l}{n\pi a} = \frac{8\lambda l^3}{a\, n^4\pi^4} \quad \text{…………………………………………..(9)}$$

Substituting (9) in (I)

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8\lambda l^3}{an^4\pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

RESULT $y(x,t) = \sum_{n=odd}^{\infty} \frac{8\lambda l^3}{an^4\pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$

8. A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v where $v = \begin{cases} \frac{cx}{l} in \ \ 0 < x < l \\ \frac{c(2l-x)}{l} in \ \ 0 < x < 2l \end{cases}$

x being the distance from one end point. Find the displacement of the string at any time.

[AU / DEC -2010]

(OR)

PUT C=1

A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v where $v = \begin{cases} \frac{x}{l} in \ \ 0 < x < l \\ \frac{(2l-x)}{l} in \ \ 0 < x < 2l \end{cases}$

x being the distance from one end point. Find the displacement of the string at any time.

[AU A/M -2016]

Solution: The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary conditions are

(i) $y(0,t)=0, \forall t>0$

(ii) $y(2l,t)=0, \forall t>0$

(iii). $y(x,0)=0, 0<x<l$

(iv). $\frac{\partial y}{\partial t}(x,0)= f(x), 0<x<2l.$

The correct solution on the boundary condition is

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat).........(1)$$

Applying condition (i) in (1), we get

$$y(0,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$

$$c_1=0,\ (c_3\cos pat+c_4\sin pat)\neq 0$$

Substituting $c_1=0$ *in eqn* (1)

$$y(x,t)=c_2\sin px(c_3\cos pat+c_4\sin pat) \quad ....(2)$$

Applying condition (ii) in (2), we get

$$y(2l,t)=c_2\sin 2pl(c_3\cos pat+c_4\sin pat)=0$$

Here, $(c_3\cos pat+c_4\sin pat)\neq 0$

Therefore, either $c_2=0$ (or) $\sin 2pl=0$

Suppose, we take $c_2=0$ and already we have $c_1=0$ then we get a trivial Solution.

we consider $c_2\neq 0$ *and* $\sin 2pl=0$

$$\sin p2l=0,\ 2pl=\sin n\pi,\ 2pl=n\pi \quad \Rightarrow p=\frac{n\pi}{2l}$$

Substituting $p=\frac{n\pi}{2l}$ **in (2) we get**

$$y(x,t)=c_2\sin\frac{n\pi x}{2l}(c_3\cos\frac{n\pi at}{2l}+c_4\sin\frac{n\pi at}{2l}) \quad ....(3)$$

Applying condition (iii) in equation (3) we get

$$y(x,0)=c_2\sin\frac{n\pi x}{2l}c_3=0$$

Here $\sin\frac{n\pi x}{2l}\neq 0$ [$\because$ It is defined for all x]

$c_2\neq 0$ [$\because$ $c_2=0$ we already explained]

Therefore $c_3 = 0$

Substitute $C_3 = 0$ in equation (3) we get

$$y(x,t) = c_2 c_4 \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$$

$$= c_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}, \quad c_2 c_4 = c_n$$

The most general solution we get

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{I})$$

Partially diff w. r. to 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2l}\left(\frac{n\pi a}{2l}\right)\cos\frac{n\pi at}{2l} \text{---------------------------}\ldots\ldots\ldots\ldots\ldots(4)$$

Now we apply condition (iv) in (4) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2l}\frac{n\pi a}{2l} = \begin{cases} \frac{cx}{l} in & 0 < x < l \\ \frac{c}{l}(2l - x) in & l < x < 2l \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Now to find c_n expand f(x) in a half-range Fourier series , we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l} = \begin{cases} \frac{cx}{l} in & 0 < x < l \\ \frac{c}{l}(2l - x) in & l < x < 2l \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6)$$

From equations (5) and (6) we get

$$\sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{2l}\right)\sin\frac{n\pi x}{2l} = \sum_{n=1}^{\infty} b_n Sin\frac{n\pi x}{2l}$$

Equating like co-efficients, we get

$$c_n = b_n\left(\frac{n\pi a}{2l}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

To find B_n expand f(x) in a Half range Fourier sine series in the interval (0, 2l)

$$b_n = \frac{2}{(2l)}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}dx$$

$$b_n = \frac{1}{l}\left[\int_0^{l}\frac{cx}{l}\sin\frac{n\pi x}{2l}dx + \int_l^{2l}\frac{c}{l}(2l - x)\sin\frac{n\pi x}{2l}dx\right]$$

$$= \frac{c}{l^2}\left[\int_0^{l} x\sin\frac{n\pi x}{2l}dx + \int_l^{2l}(2l - x)\sin\frac{n\pi x}{2l}dx\right]$$

$$= \frac{c}{l^2}\left\{\left[x\left[\frac{-\cos\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)}\right]-(1)\left[\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right]\right]_0^l+\left[(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right)\right]-(1)\left[\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right]_l^{2l}\right\}$$

$$= \frac{c}{l^2}\left[-x\left(\frac{2l}{n\pi}\right)\cos\frac{n\pi x}{2l}+\left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi x}{2l}\right]_0^l$$

$$-\frac{c}{l^2}\left[(2l-x)\left(\frac{2l}{n\pi}\right)\left(\cos\frac{n\pi x}{2l}\right)+\left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi x}{2l}\right]_l^{2l}$$

$$= \frac{c}{l^2}\left[\left(\frac{2l^2}{n\pi}\cos\left(\frac{n\pi l}{2l}\right)+\left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right)-(-0+0)\right]$$

$$-\frac{c}{l^2}\left[(0-0)-\left(\frac{2l^2}{n\pi}\cos n\pi+\left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{c}{l^2}\left[-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2l}+\left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi}{2}+\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}+\left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]$$

$$= \frac{c}{l^2}\left[2\left(\frac{2l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]=\frac{8c}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = c_n\frac{n\pi a}{2l}$$

$$c_n = \frac{2l}{n\pi a}B_n = \frac{2l}{n\pi a}\frac{c}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{16\,cl}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

$\therefore$ (5) $\Rightarrow$ $$y(x,t) = \sum_{n=1}^{\infty}\frac{16\,cl}{n^3\pi^3 a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$$

$$y(x,t) = \frac{16\,cl}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}\,..$$

RESULT: $y(x,t) = \frac{16\,cl}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}$

9. A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity given by

$$v = \begin{cases} cx & if\ 0 \le x \le \frac{l}{2} \\ c(l-x) & if\ \frac{l}{2} \le x \le l \end{cases}$$

Find the displacement function y (x,t). **[AU M/J 2007, N/D 2010]**

Solution: The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary conditions are

(i) $y(0,t) = 0, \forall t > 0$

(ii) $y(l,t) = 0, \forall t > 0$

(iii). $y(x,0) = 0, 0 < x < l$

(iv). $\frac{\partial y}{\partial t}(x,0) = f(x), 0 < x < l.$

The correct solution on the boundary condition is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat) \quad \text{.........(1)}$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1 (c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 = 0, \ (c_3 \cos pat + c_4 \sin pat) \neq 0$$

Substituting $c_1 = 0$ *in eqn* (1)

$$y(x,t) = c_2 \sin px (c_3 \cos pat + c_4 \sin pat) \quad \text{…………………(2)}$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl (c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0$ *and* $\sin pl = 0$

$$\sin pl = 0, \ pl = \sin n\pi, \ pl = n\pi \quad \Rightarrow p = \frac{n\pi}{l}$$

Substituting $p = \frac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}\right) \quad \text{…………………………(3)}$$

Applying condition (iii) in equation (3) we get

$$y(x,0) = c_2 \sin \frac{n\pi x}{l} c_3 = 0$$

Here $\sin\frac{n\pi x}{l} \neq 0$ [$\because$ It is defined for all x]

$c_2 \neq 0$ [$\because$ $c_2 = 0$ we already explained]

Therefore $c_3 = 0$

Substitute $C_3 = 0$ in equation (3) we get

$$y(x,t) = c_2 c_4 \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l},\ c_2 c_4 = c_n$$

The most general solution we get

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad \text{.................................(I)}$$

Partially diff w. r. to 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\frac{n\pi a}{l}\cos\frac{n\pi at}{l} \quad \text{---------------------------.....................(4)}$$

Now we apply condition (iv) in (4) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\frac{n\pi a}{l} = \begin{cases} cx & if\ 0 \le x \le \frac{l}{2} \\ c(l-x) & if\ \frac{l}{2} \le x \le l \end{cases} \quad \text{.......................(5)}$$

Now to find c_n expand f(x) in a half-range Fourier series , we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \begin{cases} cx & if\ 0 \le x \le \frac{l}{2} \\ c(l-x) & if\ \frac{l}{2} \le x \le l \end{cases} \quad \text{..............................(6)}$$

From equations (5) and (6) we get

$$\sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n in\frac{n\pi x}{l}$$

Equating like co-efficients, we get

$$c_n = b_n\left(\frac{n\pi a}{l}\right) \quad \text{...(7)}$$

$$b_n = \frac{2}{l}\left[\int_0^{l/2} cx\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} c(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2c}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2c}{l}\left\{\left[x\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right] + (1)\left[\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right] - (1)\left[\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_{l/2}^{l}\right\}$$

$$= \frac{2c}{l}\left[-\left(\frac{l^2}{2n\pi}\right)\cos\frac{n\pi}{2} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]$$

$$+ \frac{2c}{l}\left[0 - \left(-\frac{l^2}{2n\pi}\right)\left(\cos\frac{n\pi}{2}\right) - \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]$$

$$= \frac{2c}{l}\left[2\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right] = \frac{4lc}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$c_n = \frac{l}{n\pi a}b_n = \frac{l}{n\pi a}\frac{4lc}{n^2\pi^2}\sin\frac{n\pi}{2} = \frac{4cl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}$$

$$\therefore \text{(I)} \Rightarrow \quad y(x,t) = \sum_{n=1}^{\infty}\frac{4cl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,t) = \frac{4cl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} ..$$

RESULT: $y(x,t) = \frac{4cl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$

10. If a string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0\sin^3\frac{\pi x}{l},\ 0 < x < l$, determine the transverse displacement (or) displacement of a point distant x from one end at time't'.

Solution: **[AU N/D 2008/NOV-2013,2014]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary conditions are

$(i)\ y(0,t) = 0, \forall t > 0$

(ii) $y(l,t) = 0, \forall t > 0$

(iii). $y(x,0) = 0, 0 < x < l$

$$\text{(iv)}.\ \frac{\partial y}{\partial t}(x,0)=\lambda x(l-x)=f(x),0<x<l.$$

The correct solution on the boundary condition is

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat)..........(1)$$

Applying condition (i) in (1), we get

$$y(0,t)=c_1(c_3\cos pat+c_4\sin pat)=0$$

$$c_1=0\ ,\ (c_3\cos pat+c_4\sin pat)\neq 0$$

Substituting $c_1=0$ $in\ eqn\,(1)$

$$y(x,t)=c_2\sin px\,(c_3\cos pat+c_4\sin pat)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Applying condition (ii) in (2), we get

$$y(l,t)=c_2\sin pl\,(c_3\cos pat+c_4\sin pat)=0$$

Here, $(c_3\cos pat+c_4\sin pat)\neq 0$

Therefore, either $c_2=0$ (or) $\sin pl=0$

Suppose, we take $c_2=0$ and already we have $c_1=0$ then we get a trivial Solution.

we consider $c_2\neq 0$ and $\sin pl=0$

$$\sin pl=0\ ,\ pl=\sin n\pi\ ,\ pl=n\pi\quad\Rightarrow p=\frac{n\pi}{l}$$

Substituting $p=\frac{n\pi}{l}$ **in (2) we get**

$$y(x,t)=c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi at}{l}+c_4\sin\frac{n\pi at}{l}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Applying condition (iii) in equation (3) we get

$$y(x,0)=c_2\sin\frac{n\pi x}{l}c_3=0$$

Here $\sin\frac{n\pi x}{l}\neq 0$ [∵ It is defined for all x]

$c_2\neq 0$ [∵ $c_2=0$ we already explained]

Therefore $c_3=0$

Substitute $C_3=0$ in equation (3) we get

$$y(x,t)=c_2c_4\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l},\ c_2c_4 = c_n$$

The most general solution we get

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \qquad \text{..............................(I)}$$

Partially diff w. r. to 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\frac{n\pi a}{l}\cos\frac{n\pi at}{l} \qquad \text{--------------------------................(4)}$$

Now we apply condition (iv) in (4) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\frac{n\pi a}{l} = v_o \sin^3\frac{\pi x}{l} \qquad \text{..................................(5)}$$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = v_0 \sin^3\frac{\pi x}{l}$$

$$b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + \ldots = \frac{v_0}{4}\left[3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\right]$$

Equating the co-efficient on both sides

$$b_1 = \frac{3v_0}{4}, b_3 = -\frac{v_0}{4}\ b_2 = b_4 = b_5 = \ldots = 0$$

$$\therefore\ b_1 = c_1\frac{\pi a}{l} = \frac{3v_0}{4}$$

$$c_1 = \frac{l}{\pi a}\left[\frac{3v_0}{4}\right]$$

$$b_3 = c_3\left(\frac{3\pi a}{l}\right) = -\frac{v_0}{4}$$

$$c_3 = \frac{l}{3\pi a}\left[-\frac{v_0}{4}\right] = -\frac{lv_0}{12\pi a}$$

From equation (I), we get

$$y(x,t) = C_1\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} + C_3\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}$$

RESULT :

$$y(x,t) = \frac{3v_0 l}{4\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{v_0 l}{12\pi a}\sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l}$$

11. A Tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At the time t=0 , the string is given by a shape defined by $y(x,0) = \begin{cases} \frac{2kx}{l} & if\ 0 \le x \le \frac{l}{2} \\ 2k - \frac{2kx}{l} & if\ \frac{l}{2} \le x \le l \end{cases}$ where k is constant

, and then released from rest . Find the displacement of any point 'x' of the string at any time t>0.

Solution: **[AU / NOV -2015]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem ,we get the following boundary &initial conditions.

(i) $y(0,t) = 0, \forall\ t \geq 0$

(ii) $y(l,t) = 0, \forall\ t \geq 0$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{(x,0)} = 0, \forall\ 0 < x < l$

(iv) $y(x,0) = \begin{cases} \frac{2kx}{l} & if\ 0 \leq x \leq \frac{l}{2} \\ 2k - \frac{2kx}{l} & if\ \frac{l}{2} \leq x \leq l \end{cases}$

The correct solution on the boundary condition is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos pat + c_4 \sin pat) \text{..........(1)}$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1\,(c_3 \cos pat + c_4 \sin pat) = 0$$

$c_1 = 0$, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Substituting $c_1 = 0\ \ in\ eqn\,(1)$

$$y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \text{ ....(2)}$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0\ \ and\ \ \sin pl = 0$

$\sin pl = 0$, $pl = \sin n\pi$, $pl = n\pi$ $\Rightarrow p = \frac{n\pi}{l}$

Substituting $p = \frac{n\pi}{l}$ **in (2) we get**

$$y(x,t)=c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi at}{l}+c_4\sin\frac{n\pi at}{l}\right)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(3)}$$

Applying condition (iii) in equation (3) we get

$$y(x,0)=c_2\sin\frac{n\pi x}{l}c_3=0$$

Here $\sin\frac{n\pi x}{l}\neq 0$ [$\because$ It is defined for all x]

$c_2\neq 0$ [$\because$ $c_2=0$ we already explained]

Therefore $c_3=0$

Substitute $C_3 = 0$ in equation (3) we get

$$y(x,t)=c_2c_4\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$=c_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l},\qquad \text{where}\quad c_2c_4=C_n$$

The most general solution we get

$$y(x,t)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

Partially diff w. r. to 't' we get

$$\frac{\partial y}{\partial t}(x,t)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\text{-----------------------------(4)}$$

Now we apply condition (iv) in (4) we get

$$\frac{\partial y}{\partial t}(x,0)=\sum_{n=1}^{\infty}C_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l}=\begin{cases}\frac{2kx}{l} & \text{if } 0\le x\le\frac{l}{2}\\ 2k-\frac{2kx}{l} & \text{if } \frac{l}{2}\le x\le l\end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(5)}$$

Now to find c_n expand $\begin{cases}\frac{2kx}{l} & \text{if } 0\le x\le\frac{l}{2}\\ 2k-\frac{2kx}{l} & \text{if } \frac{l}{2}\le x\le l\end{cases}$ in a half-range Fourier series , we get

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}=\begin{cases}\frac{2kx}{l} & \text{if } 0\le x\le\frac{l}{2}\\ 2k-\frac{2kx}{l} & \text{if } \frac{l}{2}\le x\le l\end{cases}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(6)}$$

From equations (5) and (6) we get

$$\sum_{n=1}^{\infty} c_n\left(\frac{n\pi a}{l}\right)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n in\frac{n\pi x}{l}$$

Equating like co-efficients, we get

$$C_n = b_n\left(\frac{n\pi a}{l}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$b_n = \frac{2}{l}\left[\int_0^{l/2}\left(\frac{2kx}{l}\right)\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}\left(2k - \frac{2kx}{l}\right)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2k}{l^2}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2k}{l^2}\left\{\left[x\left[\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right] + (1)\left[\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^{l/2} + \left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left[\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_{l/2}^{l}\right\}$$

$$= \frac{2k}{l^2}\left[-\left(\frac{l^2}{2n\pi}\right)\cos\frac{n\pi}{2} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]$$

$$+\frac{2k}{l^2}\left[0 - \left(-\frac{l^2}{2n\pi}\right)\left(\cos\frac{n\pi}{2}\right) - \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right]$$

$$= \frac{2k}{l^2}\left[2\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{2}\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore \text{(I)} \Rightarrow \quad y(x,t) = \sum_{n=1}^{\infty}\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

RESULT: $y(x,t) = \sum_{n=1}^{\infty}\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$]

12. If a string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$,, $0 < x < l$, determine the transverse displacement (or) displacement of a point distant x from one end at time‘t’.

Solution: **[AU N/D 2016]**

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem we get the following boundary conditions are

(i) $y(0,t) = 0, \forall t > 0$

(ii) $y(l,t) = 0, \forall t > 0$

(iii). $y(x,0) = 0, 0 < x < l$

(iv). $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin\left(\dfrac{3\pi x}{l}\right)\cos\left(\dfrac{\pi x}{l}\right),\ 0 < x < l$

The correct solution on the boundary condition is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos pat + c_4 \sin pat)..........(1)$$

Applying condition (i) in (1), we get

$$y(0,t) = c_1\,(c_3 \cos pat + c_4 \sin pat) = 0$$

$$c_1 = 0\ ,\ (c_3 \cos pat + c_4 \sin pat) \neq 0$$

Substituting $c_1 = 0$ *in eqn* (1)

$$y(x,t) = c_2 \sin px\,(c_3 \cos pat + c_4 \sin pat) \quad ....(2)$$

Applying condition (ii) in (2), we get

$$y(l,t) = c_2 \sin pl\,(c_3 \cos pat + c_4 \sin pat) = 0$$

Here, $(c_3 \cos pat + c_4 \sin pat) \neq 0$

Therefore, either $c_2 = 0$ (or) $\sin pl = 0$

Suppose, we take $c_2 = 0$ and already we have $c_1 = 0$ then we get a trivial Solution.

we consider $c_2 \neq 0$ *and* $\sin pl = 0$

$\sin pl = 0$, $pl = \sin n\pi$, $pl = n\pi$ $\Rightarrow p = \dfrac{n\pi}{l}$

Substituting $p = \dfrac{n\pi}{l}$ **in (2) we get**

$$y(x,t) = c_2 \sin\frac{n\pi x}{l}\,(c_3 \cos\frac{n\pi at}{l} + c_4 \sin\frac{n\pi at}{l}) \quad \mathbf{..............................(3)}$$

Applying condition (iii) in equation (3) we get

$$y(x,0) = c_2 \sin\frac{n\pi x}{l} c_3 = 0$$

Here $\sin\dfrac{n\pi x}{l} \neq 0$ [$\because$ It is defined for all x]

$c_2 \neq 0$ [$\because$ $c_2 = 0$ we already explained]

Therefore $c_3 = 0$

Substitute $C_3 = 0$ in equation (3) we get

$$y(x,t) = c_2 c_4 \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l},\ c_2 c_4 = c_n$$

The most general solution we get

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \quad \text{.............................(I)}$$

Partially diff w. r. to 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\frac{n\pi a}{l}\cos\frac{n\pi at}{l} \quad \text{---------------------------................(4)}$$

Now we apply condition (iv) in (4) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\frac{n\pi a}{l} = v_o \sin^3\frac{\pi x}{l} \quad \text{...................................(5)}$$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = v_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$$

$$b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + ... = \frac{v_0}{2}\left[\sin\left(\frac{4\pi x}{l}\right) + \sin\left(\frac{2\pi x}{l}\right)\right]$$

Equating the co-efficient on both sides

$$\therefore\ b_1 = \frac{v_0 l}{2\pi a}$$

$$c_1 = \left[\frac{v_0}{2}\right]$$

$$B_3 = B_2 = 0$$

$$c_3 = c_2 = 0$$

$$\therefore\ C_4 = \frac{v_0}{2}$$

$$B_4 = \left[\frac{v_0 l}{8\pi a}\right]$$

From equation (I), we get

$$y(x,t) = \frac{v_0 l}{2\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{v_0 l}{8\pi a}\ \sin\frac{4\pi x}{l}\sin\frac{4\pi at}{l}$$

RESULT :

$$y(x,t) = \frac{v_0 l}{2\pi a}\sin\frac{\pi x}{l}\sin\frac{\pi at}{l} - \frac{v_0 l}{8\pi a}\ \sin\frac{4\pi x}{l}\sin\frac{4\pi at}{l}$$

3. HORIZONTALLY INFINITE PLATE

KEY WORDS (i) An infinitely Plate (iii) A long rectangular plate (ii) short edge $x = 0$

13. An infinitely long rectangular plate with insulated surface is 10cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x = 0$ is kept at temperature given by $u = \begin{cases} 20y & , \; for \; 0 \le y \le 5 \\ 20(10-y), & for \; 5 \le y \le 10 \end{cases}$ find the steady state temperature distribution in the plate **[AU / MAY -2014]**

Solution:

The two dimensional heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ----------- (*)

From the given problem we have the following boundary conditions

$(i)\;\; u(x,0) = 0 \;\; for \;\; all \;\; x$

$(ii)\;\; u(x,l) = 0 \;\; for \;\; all \;\; x$

$(iii)\;\; u(\infty, y) = 0 \;\; for \;\; x \to \infty, u \to 0$

$$(iv)\;\; u(0,y) = \begin{cases} 2l\,y & , \;\; for \;\; 0 \le y \le \dfrac{l}{2} \\ 2l\,(l-y), & for \;\; \dfrac{l}{2} \le y \le l \end{cases}$$

The correction solution of (*) which satisfies the first three boundary conditions is

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos py + D\sin py\right) \text{ --------------- } (1)$$

Applying condition (i) in (1) we get,

$$u(x,0) = \left(Ae^{px} + Be^{-px}\right)(C) = 0$$

$C = 0 \qquad \because \left(Ae^{px} + Be^{-px}\right) \ne 0$ (it is defined for all x)

Substituting C=0 in (i) we get,

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin py\right) \text{ --------------- } (2)$$

Applying condition (ii) in (2) we get,

$$u(x,l) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin pl\right) = 0$$

$\because \left(Ae^{px} + Be^{-px}\right) \ne 0 \; it \;\; is \;\; defined \;\; for \; all \;\; x$

$D \ne 0 \qquad \because \; if \;\; D = 0, alredy \; C = 0 \;\; Then \;\; we \;\; get \;\; trivial \;\; solution$

$\therefore \;\; \sin pl = 0 \qquad \sin n\pi = 0$

$$p = \frac{n\pi}{l}$$

Substituting $p = \frac{n\pi}{l}$ **in (2) we get,**

$$u(x,y) = \left(Ae^{\frac{n\pi x}{l}} + Be^{-\frac{n\pi x}{l}} \right) \left(D\sin\frac{n\pi y}{l} \right) \quad ---------------- \quad (3)$$

Applying condition(iii) in (3) we get,

$$u(\infty, y) = \left(Ae^{\infty} + Be^{-\infty} \right) \left(D\sin\frac{n\pi y}{l} \right) = 0$$

$$\sin\frac{n\pi y}{l} \neq 0 \quad \text{it is defined for all y}$$

$$D \neq 0 \quad \text{(if D=0 already explained)}$$

As $y \to \infty, u \to 0$ (condition (iii))

This is possible only when $A = 0$ $\quad \because \; if \;\; B = 0 \;\; we \;\; get \; u \to \infty$

Substituting A=0 in (3) we get,

$$u(x,y) = \left(Be^{-\frac{n\pi x}{l}} \right) \left(D\sin\frac{n\pi y}{l} \right)$$

$$= \left(c_n \, e^{-\frac{n\pi x}{l}} \sin\frac{n\pi y}{l} \right), \qquad c_n = BD$$

The most general solution can be written as

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{l} \, e^{-\frac{n\pi x}{l}} \quad ------------- \quad (4)$$

Applying condition (iv) in (4) we get,

$$u(0,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{l} = f(y) \qquad ------------- \quad (5)$$

To find c_n expand f(y) in a half range sine series

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{l} \; ------------------ \; (6)$$

$$c_n = b_n \quad using \; (5) \; \& \; (6) \; -------------- \; (7)$$

Now, $b_n = \frac{2}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}dy$

$$= \frac{2}{l}\left\{ \int_0^{\frac{l}{2}} 2ly \, \sin\frac{n\pi y}{l} dy + \int_{\frac{l}{2}}^{l} 2l(l-y) \, \sin\frac{n\pi y}{l} \, dy \right.$$

$$= 4\left(\left\{ y\left(\frac{-\cos\frac{n\pi y}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi y}{l}}{\frac{n^2\pi^2}{l^2}}\right) \right\}_0^{\frac{l}{2}} + \left\{ (l-y)\left(\frac{-\cos\frac{n\pi y}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi y}{l}}{\frac{n^2\pi^2}{l^2}}\right) \right\}_{\frac{l}{2}}^{l} \right.$$

$$= 4\left(\left\{ -\frac{l}{2}\left(\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}\right) + 1\left(\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right) \right\} + \left\{ -(l-l)\left(\frac{\cos n\pi}{\frac{n\pi}{l}}\right) - \left(\frac{-\sin n\pi}{\frac{n^2\pi^2}{l^2}}\right) - \left(-(l-\frac{l}{2})\right)\left(\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}\right) \right.\right.$$

$$\left. - \left(\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right) \right\}$$

$$= 4\left\{ -\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} \right\}$$

$$b_n = 4\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$c_n = \left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] \text{-------------------} (8)$$

Substituting (8) in (4) we get,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{l} e^{\frac{-n\pi x}{l}}$$

Replace l by 10 we get,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{10} e^{\frac{-n\pi x}{10}}$$

RESULT: $u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{10} e^{\frac{-n\pi x}{10}}$

14. An infinitely long rectangular plate with insulated surface is 20cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x = 0$ is kept at temperature given by $u = \begin{cases} 10y, & for\ 0 \le y \le 10 \\ 10(20-y), & for\ 10 \le y \le 20 \end{cases}$ find the steady state temperature distribution in the plate **[AU A/ M -2017]**

Solution: $l = 20, \frac{l}{2} = 10$

The two dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ------------ (*)

From the given problem we have the following boundary conditions

$(i)\ \ u(x,0)=0 \quad for \ \ all \ \ x$

$(ii)\ \ u(x,l)=0 \quad for \ \ all \ \ x$

$(iii)\ \ u(\infty,y)=0 \quad for \quad x\to\infty, u\to 0$

$$(iv)\ \ u(0,y)=\begin{cases}\frac{l}{2}y & , \ for\ 0\le y\le\frac{l}{2}\\ \frac{l}{2}(l-y)\ , & for\ \frac{l}{2}\le y\le l\end{cases}$$

The correction solution of (*) which satisfies the first three boundary conditions is

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(C\cos py+D\sin py\right) \text{ --------------- } (1)$$

Applying condition (i) in (1) we get,

$$u(x,0)=\left(Ae^{px}+Be^{-px}\right)(C)=0$$

$$C=0 \qquad \because \left(Ae^{px}+Be^{-px}\right)\neq 0 \text{ (it is defined for all x)}$$

Substituting C=0 in (i) we get,

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin py\right) \text{ --------------- } (2)$$

Applying condition (ii) in (2) we get,

$$u(x,l)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin pl\right)=0$$

$$\because \left(Ae^{px}+Be^{-px}\right)\neq 0\ it\ \ is\ \ \ defined\ \ \ for\ all\ \ \ x$$

$$D\neq 0 \qquad \because \ \ if \ \ \ D=0, alredy\ C=0\ \ Then \ \ \ we\ \ get\ \ trivial\ \ solution$$

$$\therefore \ \ \sin pl=0 \qquad \sin n\pi=0$$

$$p=\frac{n\pi}{l}$$

Substituting $p=\frac{n\pi}{l}$ **in (2) we get,**

$$u(x,y)=\left(Ae^{\frac{n\pi x}{l}}+Be^{-\frac{n\pi x}{l}}\right)\left(D\sin\frac{n\pi y}{l}\right) \text{ --------------- } (3)$$

Applying condition(iii) in (3) we get,

$$u(\infty,y)=\left(Ae^{\infty}+Be^{-\infty}\right)\left(D\sin\frac{n\pi y}{l}\right)=0$$

$\sin\frac{n\pi y}{l} \neq 0$ it is defined for all y

$D \neq 0$ (if D=0 already explained)

As $y \to \infty, u \to 0$ (condition (iii))

This is possible only when $A = 0$ $\because$ *if* $B = 0$ *we get* $u \to \infty$

Substituting A=0 in (3) we get,

$$u(x,y) = \left(Be^{-\frac{n\pi x}{l}}\right)\left(D\sin\frac{n\pi y}{l}\right)$$

$$= \left(c_n\, e^{-\frac{n\pi x}{l}} \sin\frac{n\pi y}{l}\right), \quad c_n = BD$$

The most general solution can be written as

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{l}\, e^{-\frac{n\pi x}{l}} \text{ ------------- } (4)$$

Applying condition (iv) in (4) we get,

$$u(0,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{l} = f(y) \text{ ------------- } (5)$$

To find c_n expand f(y) in a half range sine series

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{l} \text{ ------------------ } (6)$$

$$c_n = b_n \quad using\ (5)\ \&\ (6) \text{ --------------- } (7)$$

Now, $b_n = \frac{2}{l}\int_0^l f(y)\sin\frac{n\pi y}{l}\,dy$

$$= \frac{2}{l}\left\{\int_0^{\frac{l}{2}} \frac{l}{2}\, y \sin\frac{n\pi y}{l}\,dy + \int_{\frac{l}{2}}^{l} \frac{l}{2}(l-y)\sin\frac{n\pi y}{l}\,dy\right\}$$

$$= \left\{y\left(\frac{-\cos\frac{n\pi y}{l}}{\frac{n\pi}{l}}\right) - 1\left(\frac{-\sin\frac{n\pi y}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{\frac{l}{2}} + \left\{(l-y)\left(\frac{-\cos\frac{n\pi y}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi y}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{\frac{l}{2}}$$

$$=\left\{-\frac{l}{2}\left(\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}\right)+1\left(\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right)\right\}+\left\{-(l-l)\left(\frac{\cos n\pi}{\frac{n\pi}{l}}\right)-\left(\frac{-\sin n\pi}{\frac{n^2\pi^2}{l^2}}\right)-\left(-(l-\frac{l}{2})\right)\left(\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}\right)\right.$$

$$\left.-\left(\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right)\right\}$$

$$=\left\{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$b_n=\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$c_n=\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]------------------(8)$$

Substituting (8) in (4) we get,

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{l}e^{\frac{-n\pi x}{l}}$$

Replace l by 10 we get,

$$u(x,y)=\sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{10}e^{\frac{-n\pi x}{10}}$$

RESULT: $u(x,y)=\sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{10}e^{\frac{-n\pi x}{10}}$

15. An infinitely long metal plate in the form of an area is enclosed between the lines $y=0$ *and* $y=\pi$ for $x>0$. The temperature is zero along the edges $y=0$ *and* $y=\pi$ and at infinity. If the edges $x=0$ is kept at a constant temperature $T_0{}^0C$,find the steady state temperature at any point of the plate? **[AU / DEC -2009]**

Solution:

The two dimensional heat flow equation is $\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}=0$ ------------ (*)

From the given problem we have the following boundary conditions are

$(i)\ \ u(x,0)=0\ \ for\ \ all\ \ x$

$(ii)\ \ u(x,\pi)=0\ \ for\ \ all\ \ x$

$$(iii)\quad u(\infty, y) = 0 \quad for \quad x \to \infty, u \to 0$$

$$(iv)\quad u(0, y) = T_0 \;, 0 \le y \le \pi$$

The correction solution of (*) which satisfies the first three boundary conditions is

$$u(x, y) = \left(Ae^{\,px} + Be^{\,-px}\right)\left(C\cos py + D\sin py\right) \;---------------\; (1)$$

Applying condition (i) in (1) we get,

$$u(x,0) = \left(Ae^{\,px} + Be^{\,-px}\right)(C) \;= 0$$

$$C = 0 \qquad \because \left(Ae^{\,px} + Be^{\,-px}\right) \ne 0$$

Substituting C=0 in (1) we get,

$$u(x, y) = \left(Ae^{\,px} + Be^{\,-px}\right)\left(D\sin py\right) \;---------------\; (2)$$

Applying condition (ii) in (2) we get,

$$u(x, \pi) = \left(Ae^{\,px} + Be^{\,-px}\right)\left(D\sin p\pi\right) \;= 0$$

$$D \ne 0 \qquad \because \; if \;\; D = 0 \;\; we \;\; get\, trivial \;\; solution$$

$$\therefore\; \sin p\pi = 0 \quad or \quad p\pi = n\pi$$

$$p = n$$

Substituting p = n in (2) we get,

$$u(x, y) = \left(Ae^{\,nx} + Be^{\,-nx}\right)\left(D\sin ny\right) \;---------------\; (3)$$

Applying condition (iii) in (3) we get,

$$u(\infty, y) = \left(Ae^{\,\infty} + Be^{\,-\infty}\right)\left(D\sin ny\right) \;= 0$$

As $y \to \infty,\, u \to 0$ (condition (iii))

This is possible only when $A = 0 \qquad \because \; if \;\; B = 0 \;\; we \;\; get \;\; u \to \infty$

Substituting A=0 in (3) we get,

$$u(x, y) = \left(Be^{\,-nx}\right)\left(D\sin ny\right)$$

$$= \left(c_n\, e^{\,-nx}\sin ny\right), \qquad c_n = BD$$

The most general solution can be written as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin ny \; e^{\,-nx} \;-----\; (4)$$

Applying condition (iv) in (4) we get,

$$u(0, y) = \sum_{n=1}^{\infty} c_n \sin ny = T_0 = f(y) \;------------------\; (5)$$

To find c_n expand T_0 in a sine series in $(0, \pi)$

$$T_0 = \sum_{n=1}^{\infty} b_n \sin ny -------------------(6)$$

$$c_n = b_n \quad using\ (5)\ \&\ (6) ---------------(7)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(y)\sin ny\, dy$$

$$= \frac{2}{\pi}\int_0^{\pi} T_0\ \sin ny\, dy$$

$$= \frac{2T_0}{\pi}\left[\frac{-\cos ny}{n}\right]_0^{\pi}$$

$$= \frac{2T_0}{\pi}\left[\frac{-\cos n\pi + 1}{n}\right]$$

$$= \frac{2T_0}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

$$b_n = \begin{cases} 0\ , \ when\ \ n\ is\ even \\ \dfrac{4T_0}{\pi},\ \ when\ \ n\ is\ odd \end{cases}$$

$$(7) \Rightarrow c_n = \begin{cases} 0\ , \ when\ \ n\ is\ even \\ \dfrac{4T_0}{\pi},\ \ when\ \ n\ is\ odd \end{cases} --------------(8)$$

Substituting (8) in (4) we get,

$$u(x,y) = \sum_{n=1,3,5,...}^{\infty} \frac{4T_0}{\pi}\sin ny\, e^{-nx}$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{4T_0}{\pi}\sin(2n-1)y\, e^{-(2n-1)x}$$

RESULT: $u(x,y) = \sum_{n=1}^{\infty} \frac{4T_0}{\pi}\sin(2n-1)y\, e^{-(2n-1)x}$

16. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angle to them. The breadth of this edge $x = 0$ *is* π, this end is maintained at temperature as $u = k(\pi y - y^2)$ at all points while the other edges are at zero temperature. Determine the temperature u(x,y) at any point of the plate in the steady state if u satisfies Laplace equation? **[AU / MAY -2010]**

Solution:

The two dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ------------ (*)

From the given problem we have the following boundary conditions

$$(i)\ \ u(x,0)=0 \ \ for\ \ all\ \ x$$

$$(ii)\ \ u(x,\pi)=0 \ \ for\ \ all\ \ x$$

$$(iii)\ \ u(\infty,y)=0 \ \ for\ \ \ x\to\infty, u\to 0$$

$$(iv)\ \ u(0,y)=k(\pi y-y^2)\ ,0\le y\le\pi$$

The correction solution of (*) which satisfies the first three boundary conditions is

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(C\cos py+D\sin py\right) \text{ --------------- } (1)$$

Applying condition (i) in (1) we get,

$$u(x,0)=\left(Ae^{px}+Be^{-px}\right)(C)\ =0$$

$$C=0 \qquad \because \left(Ae^{px}+Be^{-px}\right)\neq 0$$

Substituting C=0 in (1) we get,

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin py\right) \text{ --------------- } (2)$$

Applying condition (ii) in (2) we get,

$$u(x,\pi)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin p\pi\right)=0$$

$$D\neq 0 \qquad \because\ if\ \ D=0\ \ we\ \ get\ trivial\ \ solution$$

$$\therefore\ \sin p\pi=0 \quad or \quad p\pi=n\pi$$

$$p=n$$

Substituting p = n in (2) we get,

$$u(x,y)=\left(Ae^{nx}+Be^{-nx}\right)\left(D\sin ny\right) \text{ --------------- } (3)$$

Applying condition (iii) in (3) we get,

$$u(\infty,y)=\left(Ae^{\infty}+Be^{-\infty}\right)\left(D\sin ny\right)=0$$

As $y\to\infty,\ u\to 0$ (condition (iii))

This is possible only when $A=0 \qquad \because\ if\ \ B=0\ \ we\ \ get\ \ u\to\infty$

Substituting A=0 in (3) we get,

$$u(x,y)=\left(Be^{-nx}\right)\left(D\sin ny\right)$$

$$=\left(c_n\,e^{-nx}\sin ny\right), \quad c_n=BD$$

The most general solution can be written as

$$u(x,y)=\sum_{n=1}^{\infty} c_n \sin ny \; e^{-nx} \quad -----.. \quad (4)$$

Applying condition (iv) in (4) we get,

$$u(0,y)=\sum_{n=1}^{\infty} c_n \sin ny = k(\pi y - y^2) -------------- \quad (5)$$

To find c_n expand $k(\pi y - y^2)$ in a half range sine series in $(0,\pi)$

$$k(\pi y - y^2)=\sum_{n=1}^{\infty} b_n \sin ny -------------- \quad (6)$$

From (5) and (6) we get, $c_n = b_n$ ------------------(7)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(y)\sin ny \; dy$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} k(\pi y - y^2)\sin ny \; dy$$

$$b_n = \frac{2k}{\pi}\left(\left\{ (\pi y - y^2)\left(\frac{-\cos ny}{n}\right) - (\pi - 2y)\left(\frac{-\sin ny}{n^2}\right) + (-2)\left(\frac{\cos ny}{n^3}\right) \right\}_0^{\pi} \right.$$

$$= \frac{2k}{\pi}\left(\left\{ -(\pi^2-\pi^2)\left(\frac{\cos n\pi}{n}\right) + (\pi - 2\pi)\left(\frac{\sin n\pi}{n^2}\right) - 2\left(\frac{\cos n\pi}{n^3}\right) \right\} - \left\{ 0 + (\pi - 0)\left(\frac{\sin 0}{n^2}\right) - 2\left(\frac{\cos 0}{n^3}\right) \right\} \right)$$

$$= \frac{2k}{\pi}\left\{ \left(\frac{-2\cos n\pi}{n^3}\right) + \left(\frac{2}{n^3}\right) \right\}$$

$$= \frac{4k}{n^3\pi}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0 \quad , & when \;\; n \; is \; even \\ \dfrac{8k}{n^3\pi}, & when \;\; n \; is \, odd \end{cases}$$

$$\therefore c_n = \begin{cases} 0 \quad , & when \;\; n \; is \; even \\ \dfrac{8k}{n^3\pi}, & when \;\; n \; is \, odd \end{cases}$$

Substituting this value of C_n in (4) we get,

$$u(x,y)=\sum_{n=1,3,5...}^{\infty} \frac{8k}{n^3\pi}\sin ny \; e^{-nx}$$

$$= \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)y \; e^{-(2n-1)x}$$

RESULT: $u(x,y)=\frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin(2n-1)y\ e^{-(2n-1)x}$

17. An infinitely long plate in the form of an area is enclosed between the lines $y=0\ \ and\ \ y=\pi$ for positive value of x. The temperature is zero along the edges $y=0\ \ and\ \ y=\pi$ and the edge at infinity. If the edge $x=0$ is kept at the temperature $f(y)=ky,\ 0<y<\pi$ find the steady state temperature distribution in the plate**?**

Solution: **[AU / MAY -2010]**

The two dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ------------- (*)

From the given problem we have the following boundary conditions

$(i)\ \ u(x,0)=0\ \ for\ \ all\ \ x$

$(ii)\ \ u(x,\pi)=0\ \ for\ \ all\ \ x$

$(iii)\ \ u(\infty,y)=0\ \ for\ \ x\to\infty, u\to 0$

$(iv)\ \ u(0,y)=ky, 0\le y\le\pi$

The correction solution of (*) which satisfies the first three boundary conditions is

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(C\cos py+D\sin py\right)\ \text{---------------}\ (1)$$

Applying condition (i) in (1) we get,

$$u(x,0)=\left(Ae^{px}+Be^{-px}\right)(C)=0$$

$$C=0 \qquad \because \left(Ae^{px}+Be^{-px}\right)\neq 0$$

Substituting C=0 in (1) we get,

$$u(x,y)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin py\right)\ \text{---------------}\ (2)$$

Applying condition (ii) in (2) we get,

$$u(x,\pi)=\left(Ae^{px}+Be^{-px}\right)\left(D\sin p\pi\right)=0$$

$$D\neq 0 \qquad \because\ if\ \ D=0\ \ we\ \ get\ trivial\ \ solution$$

$$\therefore\ \sin p\pi=0 \quad or \quad p\pi=n\pi$$

$$p=n$$

Substituting p = n in (2) we get,

$$u(x,y)=\left(Ae^{nx}+Be^{-nx}\right)\left(D\sin ny\right)\ \text{---------------}\ (3)$$

Applying condition (iii) in (3) we get,

$$u(\infty,y)=\left(Ae^{\infty}+Be^{-\infty}\right)\left(D\sin ny\right)=0$$

As $y\to\infty,\ u\to 0$ (condition (iii))

This is possible only when $A = 0 \quad \because \text{ if } B = 0 \text{ we get } u \to \infty$

Substituting A=0 in (3) we get,

$$u(x,y) = \left(Be^{-nx}\right)\left(D\sin ny\right)$$
$$= \left(c_n e^{-nx}\sin ny\right), \quad c_n = BD$$

The most general solution can be written as

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin ny \; e^{-nx} \text{ ------------- } (4)$$

Applying condition (iv) in (4) we get,

$$u(0,y) = \sum_{n=1}^{\infty} c_n \sin ny = ky = f(y) \text{ -------------- } (5)$$

To find c_n expand ky in a half range sine series in $(0,\pi)$

$$ky = \sum_{n=1}^{\infty} b_n \sin ny \text{ -------------- } (6)$$

From (5) and (6) we get, $c_n = b_n$ ------------------(7)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(y)\sin ny \; dy$$

$$= \frac{2}{\pi}\int_0^{\pi} ky \; \sin ny \; dy$$

$$= \frac{2k}{\pi}\left(\left\{ y\left(\frac{-\cos ny}{n}\right) - (1)\left(\frac{-\sin ny}{n^2}\right) \right\}_0^{\pi} \right.$$

$$= \frac{2k}{\pi}\left\{\left(\frac{-\pi\cos n\pi}{n}\right) + \left(\frac{\sin n\pi}{n^2}\right)\right\}$$

$$= \frac{2k}{\pi}\left[-\pi\cos n\pi\right]$$

$$b_n = \frac{2k}{\pi}(-1)^{n+1}$$

$$\therefore c_n = \frac{2k}{\pi}(-1)^{n+1} \text{ ---------------------- } (8)$$

Substituting (8) in (4) we get,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2k}{n}(-1)^{n+1}\sin ny \; e^{-nx}$$

\

$$u(x,y) = 2k\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin ny\ e^{-n\,x}$$

4. VERTICALLY INFINITE PLATE

KEY WORDS (i) An infinitely Plate (ii)A long rectangular plate (ii) short edge $y = 0$

18. A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge y=0 is given by $u = \begin{cases} 20\ x\ \ for\ \ 0 \le x \le 5 \\ 20(10-x)\ \ for\ 5 \le x \le 10 \end{cases}$ and all the other three edges are kept at $0^\circ C$. Find the steady state temperature at any point in the plane. **[AU / MAY -2013]**

Solution:

The two dimensional heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

From the given problem we have the following boundary conditions

$(i)\quad u(0,y) = 0\quad for\quad all\quad y$

$(ii)\quad u(l,y) = 0\quad for\quad all\quad y$

$(iii)\quad u(x,\infty) = 0\quad o < x < l$

$$(iv)\quad u(x,0) = \begin{cases} 2l\,x & ,\quad for\ \ 0 \le x \le \dfrac{l}{2} \\ 2l\,(l-x)\ , & for\ \ \dfrac{l}{2} \le x \le l \end{cases}$$

The correct solution of (1) which satisfies our boundary conditions is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Applying condition (i) in (2) we get

$$u(0,y) = (A)(Ce^{py} + De^{-py}) = 0$$

$$A = 0\ \ [\because Ce^{py} + De^{-py} \ne 0]$$

Substituting $A = 0$ **in (2), we get**

$$u(x,y) = (B\sin px)(Ce^{py} + De^{-py}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Applying condition (ii) in (2) we get,

$$u(l,y) = (B\sin pl)(Ce^{py} + De^{-py})$$

$$\because\ (Ce^{\,py} + Be^{\,-py}) \ne 0\ it\ \ is\quad defined\quad for\, all\quad y$$

$$B \neq 0 \qquad \because \ if \quad B = 0, alredy\ A = 0\ \ Then \quad we\ \ get\ \ trivial\ \ solution$$

$$\therefore \ \sin pl = 0 \qquad \sin n\pi = 0$$

$$p = \frac{n\pi}{l}$$

Substituting $p = \dfrac{n\pi}{l}$ **in (2) we get**

$$u(x,y) = \left(B\sin\frac{n\pi}{l}x\right)\left(Ce^{\frac{n\pi y}{l}} + De^{\frac{-n\pi y}{l}}\right)\ldots\ldots\ldots\ldots\ldots,,(3)$$

Applying condition (iii) in (3), we get

$$u(x,\infty) = \left(B\sin\frac{n\pi}{l}x\right)\left(Ce^{\infty} + De^{-\infty}\right) = 0$$

As $y \to \infty,\ u \to 0\ (condition(iii))$

This is possible when $C = 0\ [\because\ if\ \ D = 0\ \ we\ \ get\ u \to \infty]$

Substituting $C = 0\ in\ (4)$ **we get**

$$u(x,y) = B\sin\left(\frac{n\pi}{l}x\right).De^{-\frac{n\pi y}{l}}$$

$$u(x,y) = C_n\sin\left(\frac{n\pi}{l}x\right)e^{-\frac{n\pi y}{l}}, \quad BD = C_n$$

The most general solution can be written as

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{l}x\right)e^{-\frac{n\pi y}{l}}\ \ldots\ldots\ldots\ldots\ldots,,,(4)$$

Applying Condition (iv) in (5) we get

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}x = u(x,0) = \begin{cases} 2l\,x, & for\ \ 0 \le x \le \dfrac{l}{2} \\ 2l\,(l-x), & for\ \ \dfrac{l}{2} \le x \le l \end{cases}$$

To find c_n **, expand f(x) in a half-range Fourier sine series in** $(0,l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

From (6) and (7) we get

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

$$\therefore c_n = b_n\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Now $\qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x dx$

Here $l = 10$

$$b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx$$

$$= \frac{1}{5}\left\{\int_0^5 f(x)\sin\frac{n\pi x}{10}dx + \int_5^{10} f(x)\sin\frac{n\pi x}{10}dx\right\}$$

$$= \frac{1}{5}\left\{\int_0^5 20\sin\frac{n\pi x}{10}dx + \int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}dx\right\}$$

$$= 4\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^5 + \left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_5^{10}\right\}$$

$$= 4\left\{\left[\frac{-10}{n\pi}.5.\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} - 0 - 0\right] + \left[-0 - 0 + \frac{10}{n\pi}.5.\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\right\}$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} \text{..............................(7)}$$

Substituting (7) in (3),

$$u(x,y) = \sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{\frac{-n\pi y}{10}}$$

RESULT: $\qquad u(x,y) = \sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{\frac{-n\pi y}{10}}$

19. A long rectangular plate has its surfaces insulated and the two long sides as well as one of the short sides are maintained at $0^\circ C$. Find an expression for the steady state temperature $u(x,y)$ if the short side $y = 0$ *is* π *cm* long and is kept at $u_0^\circ C$.

Solution: **[AU / MAY -2009]**

The two dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ …………………..(1)

The boundary conditions are

(i) $u(0,y) = 0 \;\; for \;\; all \;\; 'y'$

(ii) $u(\pi,y) = 0 \;\; for \;\; all \;\; 'y'$

(iii) $u(x,\infty) = 0 \;\; i.e., \; u \to 0 \;\; when \;\; y \to \infty$

(iv) $u(x,0) = u_0 \;\; for \;\; x \; in \; (0,\pi)$

The correct solution of (1) which satisfies our boundary conditions is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Applying condition (i) in (2) we get

$$u(0,y) = (A)(Ce^{py} + De^{-py}) = 0$$

$$A = 0 \;\; [\because Ce^{py} + De^{-py} \neq 0]$$

Substituting $A = 0$ **in (2), we get**

$$u(x,y) = (B\sin px)(Ce^{py} + De^{-py}) \;\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Applying condition (ii) in (3) we get

$$u(\pi,y) = (B\sin p\pi)(Ce^{py} + De^{-py}) = 0$$

Here $B \neq 0 \;\; [\because B = 0 \; we \; get \; trivial \, solution]$

$$\therefore \sin p\pi = 0 \; or \;\; p\pi = n\pi$$

$$p = n$$

Substituting $p = n$ **in (3) we get**

$$u(x,y) = (B\sin nx)(Ce^{ny} + De^{-ny}) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Applying condition (iii) in (4), we get

$$u(x,\infty) = (B\sin nx)(Ce^{\infty} + De^{-\infty}) = 0$$

As $y \to \infty, u \to 0 \; (condition(iii))$

This is possible when $C = 0 \; [\because if \;\; D = 0 \; we \;\; get \; u \to \infty]$

S**ubstituting** $C = 0 \; in \; (4)$ **we get**

$$u(x,y) = B\sin nx.De^{-ny}$$

$$= C_n \sin nxe^{-ny}, \quad BD = C_n$$

The most general solution can be written as

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin nxe^{-ny} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Applying Condition (iv) in (5) we get

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx = u_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

To find c_n **, expand** u_0 **in a half-range Fourier sine series in** $(0,\pi)$

$$u_0 = \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

From (6) and (7) we get

$$\sum_{n=1}^{\infty} c_n \sin nx = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore c_n = b_n \text{.............................}(8)$$

Now $$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx dx$$

$$= \frac{2u_0}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2u_0}{n\pi}\left[-\cos n\pi + \cos 0\right]$$

$$= \frac{2u_0}{n\pi}\left[(-1)^{n+1} + 1\right]$$

$$b_n = \begin{cases} 0, & when\ 'n'\ even \\ \dfrac{4u_0}{n\pi}, & when\ 'n'\ odd \end{cases}$$

$$c_n = \frac{4u_0}{n\pi}, \quad when\ 'n'\ odd$$

Substituting this value of c_n in (5) we get

$$u(x,y) = \sum_{n=1,3,5}^{\infty} \frac{4u_0}{n\pi} \sin nx e^{-ny}$$

RESULT:

$$u(x,y) = \sum_{n=1,3,5}^{\infty} \frac{4u_0}{n\pi} \sin nx e^{-ny}$$

20. A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temperature along one short edge y=0 is given by $u(x,0) = 100\sin\frac{\pi x}{8} \ \ in\ 0 < x < 8$

While the two long edges x=0 and x=8 as well as the other short edges are kept at $0^\circ C$, find the steady state temperature function $u(x,y)$ **[AU / MAY -2010]**

Solution:

The two dimensional heat flow equation is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ……………………..(1)

The boundary conditions are

(i) $u(0, y) = 0 \ \ for\ \ all\ \ 'y'$

(ii) $u(l, y) = 0 \ \ for\ \ all\ \ 'y'$

(iii) $u(x, \infty) = 0$,

(iv) $u(x,0) = 100\sin\frac{\pi x}{l} \ \ for\ \ x\ \ in\ \ (0, l)$

The correct solution of (1) which satisfies our boundary conditions is

$$u(x, y) = (A\cos px + B\sin px)\left(Ce^{py} + De^{-py}\right) \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

See problem number (12)

The most general solution can be written as

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} e^{\frac{-n\pi y}{l}} \ \ldots\ldots\ldots\ldots\ldots(3)$$

Applying Condition (iv) in (3) we get

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = 100\sin\frac{\pi x}{l}$$

$$c_1 \sin\frac{\pi x}{l} + c_2 \sin\frac{\pi x}{l} + \ldots\ldots\ldots = 100\sin\frac{\pi x}{l}$$

Equating like terms on both sides,

$$c_1 = 100, c_2 = 0,\ c_3 = c_4 = c_5 = \ldots.. = 0$$

Substituting in (3), we get

$$u(x, y) = 100\sin\frac{\pi x}{l} e^{\frac{-\pi y}{l}}$$

RESULT:

$$u(x, y) = 100\sin\frac{\pi x}{l} e^{\frac{-\pi y}{l}}$$

21. An infinitely long plane uniform plate is bounded by two parallel edges x=0 and x=l and an end at right angles to them. The breadth of this edge y=0 is l and is maintained at a $f(x) = k(lx - x^2)$. All the other three edges are at $0^\circ C$. Find the steady state temperature

temperature at any interior point of the plate. **[AU / MAY -2010, A/M 2016]**

Solution:

The two dimensional heat flow equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **…………………..(1)**

The boundary conditions are

(i) $u(0,y)=0\ \ for\ all\ 'y'$

(ii) $u(l,y)=0\ \ for\ all\ 'y'$

(iii) $u(x,\infty)=0\ ,\ for\ all\ 'y'$

(iv) $u(x,0)=k(lx-x^2)\ for\ x\ in\ (0,l)$

The correct solution of (1) which satisfies our boundary conditions is

$$u(x,y)=\left(A\cos px+B\sin px\right)\left(Ce^{py}+De^{-py}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Using problem number 2

The most general solution can be written as

$$u(x,y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{l}e^{\frac{-n\pi y}{l}}\ \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Applying Condition (iv) in (3) we get

$$u(x,0)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{l}=k(lx-x^2)\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

To find c_n , expand $f(x)$ in a half-range Fourier sine series

$$f(x)=k(lx-x^2)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{10}\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

From (4) and (5) we get

$$c_n=b_n\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$$b_n=\frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2k}{l}\left\{\left(lx-x^2\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-\left(l-2x\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+\left(-2\right)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^l$$

$$=\frac{2k}{l}\left\{\left[-0+0-\frac{2\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]-\left[0+0-\frac{2}{\frac{n^3\pi^3}{l^3}}\right]\right\}$$

$$=\frac{2k}{l}\left\{-\frac{2\cos n\pi}{\frac{n^3\pi^3}{l^3}}+\frac{2}{\frac{n^3\pi^3}{l^3}}\right\}$$

$$= \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0, & if \ n \ is \ even \\ \dfrac{8kl^2}{n^3\pi^3}, & if \ n \ is \ odd \end{cases}$$

$$\therefore \quad c_n = \begin{cases} 0, & if \ n \ is \ even \\ \dfrac{8kl^2}{n^3\pi^3}, & if \ n \ is \ odd \end{cases}$$

Substituting in (3),

$$u(x,y) = \sum_{n=1,3,5}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi x}{l} e^{\frac{-n\pi y}{l}}$$

$$= \frac{8kl^2}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} e^{\frac{-n\pi y}{l}}$$

RESULT:

$$u(x,y) = \frac{8kl^2}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} e^{\frac{-n\pi y}{l}}$$

5.TWO DIMENSIONAL HEAT FLOW EQUATION (SQUARE PLATE)

22. A square plate is bounded by the lines x=0,y=0,x=20 and y=20. Its faces are insulated, the temperature along the edge y=20 is given by x (20 - x) while the other three edges are kept at $0^\circ C$. Find the steady-state temperature distribution on the plate. **[AU / DEC -2016]**

Solution:

The equation to be solved in $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The boundary conditions are

(i) u(0,y)=0

(ii) u(20,y)=0

(iii) u(x,0)=0

(iv) u(x,20)= x(20-x)

Suitable solution is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \text{----------------------}(1)$$

Apply (i) in (1)

$$u(0,y) = A(Ce^{py} + De^{-py}) = 0$$

$$A=0 \quad \left[\because (Ce^{py} + De^{-py}) \neq 0\right]$$

Sub A=0 in (1)

$$u(x,y) = B\sin px(Ce^{py} + De^{-py}) \text{------------------------------}(2)$$

Apply (ii) in (2)

$$u(20,y) = B\sin 20p(Ce^{py} + De^{-py}) = 0$$

Sin20p=0 $\left[\because (Ce^{py} + De^{-py}) \neq 0 \text{ and } B \neq 0\right]$

Sin20p = sinnπ

20P = nπ

$$p = \frac{n\pi}{20}$$

Now, (2) becomes

$$u(x,y) = B\sin\frac{n\pi x}{20}(Ce^{\frac{n\pi y}{20}} + De^{-\frac{n\pi y}{20}}) \text{--------------------------------------}(3)$$

Apply (iii) in (3)

$$u(x,0) = B\sin\frac{n\pi x}{20}(C + D) = 0$$

$$(C + D) = 0 \qquad \left[\because \sin\frac{n\pi}{20}x \neq 0 \text{ and } B \neq 0\right]$$

$$D = -C$$

Now, (3) becomes

$$u(x,y) = B\sin\frac{n\pi x}{20}(Ce^{\frac{n\pi}{20}y} - C\ e^{-\frac{n\pi}{20}y})$$

$$u(x,y) = BC\sin\frac{n\pi x}{20}2\sinh\frac{n\pi y}{20} \qquad BC = A_n$$

$$u(x,y) = A_n\sin\frac{n\pi x}{20}2\sinh\frac{n\pi y}{20}$$

The most general form is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20} \text{ ....}(4)$$

Apply condition (iv) in (4)we get

$$u(x,20) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{20}\sinh n\pi = x(20 - x)\text{......................}(5)$$

To find B_n expand $f(x)$ in a Fourier half range sine series

$$\therefore x(20-x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20} \quad -------(6)$$

From (5) and (6) we get

$$A_n \sin hn\pi = B_n \qquad\qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$A_n = \frac{B_n}{\sin hn\pi}$$

$$B_n = \frac{2}{20}\int_0^{20} x(20-x)\sin\left(\frac{n\pi x}{20}\right)dx$$

$$= \frac{1}{10}\left[(20x-x^2)\left[\frac{-\cos\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)}\right]-(20-2x)\left[\frac{-\sin\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^2}\right]+(-2)\left[\frac{\cos\frac{n\pi x}{20}}{\left(\frac{n\pi}{20}\right)^3}\right]\right]_0^{20}$$

$$= \frac{1}{10}\left[-(20x-x^2)\left[\frac{20}{n\pi}\right]\cos\frac{n\pi x}{20}+(20-2x)\left[\left(\frac{20}{n\pi}\right)^2\sin\frac{n\pi x}{20}\right]-2\left(\frac{20}{n\pi}\right)^3\cos\frac{n\pi x}{20}\right]_0^{20}$$

$$= \frac{1}{10}\left[\left(-0+0-2\left[\frac{20}{n\pi}\right]^3\cos n\pi\right)-\left(-0+0-2\left(\frac{20}{n\pi}\right)^3\right)\right]_0^{20}$$

$$= \frac{1}{10}\left[\left(-2\left[\frac{20}{n\pi}\right]^3(-1)^n\right)+2\left(\frac{20}{n\pi}\right)^3\right]$$

$$= \frac{1}{10}\frac{8(10)^3}{n^{3.}\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{1600}{n^{3.}\pi^3}\left[1-(-1)^n\right]$$

$$= \begin{cases}\dfrac{3200}{n^3\pi^3} & if\ \ n\ is\ odd\\ 0 & if\ n\ is\ even\end{cases}$$

$$\therefore A_n = \frac{B_n}{\sin hn\pi} = \frac{3200}{n^{3.}\pi^3\sin h\,n\pi}\ \ if\ \ n\ is\ odd.$$

$$= 0\ if\ \ n\ is\ even.$$

Substitute the value of A_n in equation (5) we get

$$u(x,y)=\sum_{n=odd}^{\infty}\frac{3200}{n^{3.}\pi^3\sin h\,n\pi}\sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20}$$

$$= \frac{3200}{\pi^3} \sum_{n=odd}^{\infty} \frac{1}{n^{3.} \sin h\, n\pi} \sin \frac{n\pi x}{20} \sinh \frac{n\pi y}{20}$$

$$u(x, y) = \frac{3200}{\pi^3} \sum_{n=odd}^{\infty} \frac{1}{n^{3.} \sin h\, n\pi} \sin \frac{n\pi x}{20} \sinh \frac{n\pi y}{20}$$

RESULT:

$$u(x, y) = \frac{3200}{\pi^3} \sum_{n=odd}^{\infty} \frac{1}{n^{3.} \sin h\, n\pi} \sin \frac{n\pi x}{20} \sinh \frac{n\pi y}{20}$$

23. A Square plate is bounded by the lines x= 0 , x=a, y = 0 and y = b. Its surfaces are insulated and the temperature along y=b is kept at 100^0 C. while the temperature along other three edges are at 0^0C. Find the steady-state temperature at any point in the plate.

Solution:

[AU NOV-2014]

The equation to be solved in $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The boundary conditions are

(i) u(0,y)=0

(ii) u(a,y)=0

(iii) u(x,0)=0

(iv)u(x,b)= 100

Suitable solution is

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \text{----------------------(1)}$$

Apply (i) in (1)

$$u(0, y) = A(Ce^{py} + De^{-py}) = 0$$

A=0 $\left[\because (Ce^{py} + De^{-py}) \neq 0\right]$

Sub A=0 in (1)

$$u(x, y) = B\sin px(Ce^{py} + De^{-py}) \text{-----------------------------(2)}$$

Apply (ii) in (2)

$$u(a, y) = B\sin pa(Ce^{py} + De^{-py}) = 0$$

Sinpa=0 $\left[\because (Ce^{py} + De^{-py}) \neq 0 \text{ and } B \neq 0\right]$

Sinpa = sinnπ

Pa = nπ

$$p = \frac{n\pi}{a}$$

Now, (2) becomes

$$u(x,y) = B\sin\frac{n\pi}{a}x(Ce^{\frac{n\pi}{a}y} + De^{-\frac{n\pi}{a}y}) \text{---------------------------------}(3)$$

Apply (iii) in (3)

$$u(x,0) = B\sin\frac{n\pi}{a}x(Ce^{\frac{n\pi}{a}0} + De^{-\frac{n\pi}{a}0}) = 0$$

$$= B\sin\frac{n\pi}{a}x(C+D) = 0 \quad \because e^0 = 1$$

$$= (C+D) = 0 \qquad \left[\because \sin\frac{n\pi}{a}x \neq 0 \ and \ B \neq 0\right]$$

$$D = -C$$

Now, (3) becomes

$$u(x,y) = B\sin\frac{n\pi}{a}x(Ce^{\frac{n\pi}{a}y} - Ce^{-\frac{n\pi}{a}y})$$

$$u(x,y) = B\sin\frac{n\pi}{a}x(2C\sinh\frac{n\pi}{a}y)$$

$$u(x,y) = 2BC\sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}y \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The most general form is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}y \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Apply condition(iv),we get

$$u(x,b) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}b = 100$$

Using half range Fourier sine series ,

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x = 100 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (5) and (6) we get

$$B_n = A_n \sinh\frac{n\pi}{a}b$$

Using half range sine series ,

$$B_n = \frac{2}{a}\int_0^a 100\sin\frac{n\pi x}{a}dx$$

$$B_n = \frac{200}{a}\left(\frac{-Cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right)_0^a$$

$$B_n = \frac{200}{a}\left(\frac{-Cos\frac{n\pi a}{a}}{\frac{n\pi}{a}} - \left(\frac{-Cos\frac{n\pi 0}{a}}{\frac{n\pi}{a}}\right)\right)$$

$$B_n = \frac{200}{a}\left(\frac{-(-1)^h}{\frac{n\pi}{a}} + \left(\frac{1}{\frac{n\pi}{a}}\right)\right)$$

$$B_n = \begin{cases} \frac{400}{n\pi} & if \;\; n = odd \\ 0 & if \;\; n = even \end{cases}$$

$$A_n = \begin{cases} \frac{400}{n\pi \sinh\frac{n\pi b}{a}} & if \;\; n = odd \\ 0 & if \;\; n = even \end{cases}$$

From eqn(5) becomes

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{400}{n\pi \sinh\frac{n\pi b}{a}} \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y$$

RESULT: $u(x,y) = \sum_{n=odd}^{\infty} \frac{400}{n\pi \sinh\frac{n\pi b}{a}} \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y$

6. STEADY STATE CONDITIONS AND NON-ZERO BOUNDARY

24. The ends of A and B of a rod 10cm long have their temperature kept at 50˚ C and 100˚ C respectively until the steady state conditions prevail. The temperature of the end B is then suddenly reduced to 60˚ C and kept so while the end A is raised to 90˚ C .Find the temperature distribution function in the rod after time t. **[AU N/D 2008, 2015]**

Solution:

The temperature function u(x,t) is the solution of the one dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

When the steady state condition prevails $\frac{\partial u}{\partial t}=0$ hence $\frac{\partial^2 u}{\partial x^2}=0$

The steady state solution u(x)=ax+b

Steady state (i)

When x=0 , u(0)= b=> b=50

When x=10 , u(10)=10a+b=>100=10a+50=> a=5

Thus, u(x,0)=f(x)=5x+50

Steady state(ii)

u(x)=ax+b

When x=0 , u(0)= b=> b=90

When x=10 , u(10)=10a+90=>60=10a+90=> a=--3

Thus, u(x,0)=f(x)= $-3x+90$

u(x,t)= -3x+90+u(x,t)

Boundary and initials conditions are

(i) u(0,t)=90 for all t≥0

(ii) u(10,t)=60 for all t≥0

(iii)u(x,0)=f(x)=5x+50 for all x

Now the suitable solution is

u(x,t)= -3x+90 +(A cospx+B sinpx) $e^{-\alpha^2 p^2 t}$ ------------------------------(1)

Apply condition (i) in (1)

u(0,t)=90+ A $e^{-\alpha^2 p^2 t}$ =90

A=0 [$\because$ $e^{-\alpha^2 p^2 t}\neq 0$]

Now (1) implies u(x,t)= $-3x$ +90+ B sinpx $e^{-\alpha^2 p^2 t}$ ………………………….(2)

Apply condition (ii) in (2)

60= 50+90 +B sin(10p) $e^{-\alpha^2 p^2 t}$

=Bsin(10p) $e^{-\alpha^2 p^2 t}$ =0 [$\because$ $e^{-\alpha^2 p^2 t}\neq 0$ and B≠0]

Sin10p =Sin nπ

$$p=\frac{n\pi}{10}$$

$$u(x,t)= -3x+90 +B \sin \frac{n\pi x}{10} \; e^{-\alpha^2\left(\frac{n\pi}{10}\right)^2 t}$$

The most general solution is

$$u(x,t)= -3x+90+\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{-\alpha^2\left(\frac{n\pi}{10}\right)^2 t} \qquad \text{--------------------------------------(3)}$$

Apply condition (iii) in (3)

$$-3x+90+\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = f(x) = 5x+50$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = 5x+50+3x-90 = 8x-40$$

Which is half range sine series in the interval (0,10)

To find B_n **: The half range Fourier sine series** $\quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$B_n = b_n = \frac{2}{10}\int_0^{10}(8x-40)\sin\frac{n\pi x}{10}dx$$

$$B_n = b_n = \frac{2}{10}\left[\left((8x-40)\right)\left[-\frac{\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right]-(8)\left[-\frac{\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right]\right]_0^{10}$$

$$=\frac{2}{10}\left\{\left[40\left[-\frac{\cos n\pi}{\frac{n\pi}{10}}\right]-(8)\left[-\frac{\sin n\pi}{\left(\frac{n\pi}{10}\right)^2}\right]\right]-\left((-40)\left[-\frac{\cos 0}{\frac{n\pi}{10}}\right]-(8)\left[-\frac{\sin 0}{\left(\frac{n\pi}{10}\right)^2}\right]\right)\right\}$$

$$=\frac{2}{10}\left\{\left[40\left[-\frac{(-1)^n}{\frac{n\pi}{10}}\right]\right]-\left((-40)\left[-\frac{1}{\frac{n\pi}{10}}\right]\right)\right\}$$

$$=\frac{-80}{n\pi}\left\{(-1)^n +1\right\}$$

$$b_n = \begin{cases} 0 & if \;\; n \;\; is \;\; odd \\ \frac{-160}{n\pi} & if \;\; n \;\; is \;\; even \end{cases}$$

Now equation (3) become

u(x,t)= $-3x+90+\sum_{n=2,4,\ldots}^{\infty}\frac{-160}{n\pi}\sin\frac{n\pi x}{10}e^{-\alpha^2\left(\frac{n\pi}{10}\right)^2 t}$

RESULT: $u(x,t)=-3x+90-\frac{160}{\pi}\sum_{n=2,4,\ldots}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{10}\right)e^{\frac{-\alpha^2 n^2\pi^2 t}{100}}$

25. A bar 10cm long with insulated sides, has its ends A and B kept at 20˚ C and 40˚ C respectively until the steady state conditions prevail. The temperature of the end A is then suddenly reduced to 50˚ C and and at the same instant that at B is lowered to 10˚ C .Find the subsequent temperature at any point of the bar at any time. **[AU A/M 2018]**

Solution:

The temperature function u(x,t) is the solution of the one dimensional heat equation

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}\quad ------(1)$$

When the steady state condition prevails $\frac{\partial u}{\partial t}=0$ hence $\frac{\partial^2 u}{\partial x^2}=0$

The steady state solution u(x)=ax+b ----------(2)

(a) u(0)= 20

(b) u(l)=40

Applying condition (a) in (2) u(0)=b=20

(2) Becomes u(x)=ax+20 --------(3)

Applying condition (b) in (3)

u(l)=al+20=40

$$a=\frac{20}{l}\quad ------(4)$$

Applying (4) in (3)

$$u(x)=\frac{20}{l}x+20\quad ------(5)$$

Hence steady state is $u(x)=\frac{20}{l}x+20$

Now the temperature at A is raised at 50°C and the temperature at B is lowered to 10°C. that is the steady state is changed to unsteady state. For this unsteady state the initial temperature distribution is given by $u(x,0)=\frac{20}{l}x+20$

Boundary conditions are

(a) u(0,t)=50 for all t≥0

(b) u(l,t)=10 for all t≥0

(c) u(x,0)=20/l x+20 for all x

Now the suitable solution is

u(x,t)= (A cospx+B sinpx) $e^{-\alpha^2 p^2 t}$ ------------------------------(6)

Apply condition (a)and (b) in (6)

u(0,t)= A $e^{-\alpha^2 p^2 t}$ =50------(7), u(l,t)= (A cospl+B sinpl) $e^{-\alpha^2 p^2 t}$ =10 -----------------------(8)

from (7) and (8) it is not possible to find the values of A and B. since we have infinite number of values for A and b. therefore we split the solution u(x,t) in two parts

$u(x,t)=u_s(x)+u_t(x,t)$ ------(9)

Where $u_s(x)$ is a solution of the equation $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ and is a function of x alone and satisfying the conditions $u_s(0)=50$ and $u_s(l)=10$ and $u_t(x,t)$ is a transient solution satisfying (9) which decreases as t increases.

To find $u_s(x)$

We have $u_s(x)=a_1x+b_1$ ------(10)

Applying the conditions $u_s(0)=50\ in\ (10)$ we get

$u_s(0)=b_1=50$

$\therefore u_s(x)=a_1x+50$ -----(11)

Applying the conditions $u_s(l)=10\ in\ (11)$ we get

$\therefore u_s(l)=a_1l+50=10$

$a_1l=10-50=-40 \Rightarrow a_1=\frac{-40}{l}$ ------(12)

Substituting (12) in (11) we get, $u_s(x)=\frac{-40}{l}x+50$ ------(13)

To find $u_t(x,t)$

We assume that $u_t(x,t)$ is a transient solution of $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ and satisfying the equation

$$u(x,t)=u_s(x)+u_t(x,t)$$

$$u(x,t)-u_s(x)=u_t(x,t)\;------(14)$$

Now we have to find the boundary conditions for $u_t(x,t)$

Put $x=0\,in$ (14) we get,

$$u(0,t)-u_s(0)=u_t(0,t)\Rightarrow u_t(0,t)=50-50=0----(15)$$

Put $t=0\,in$ (14) we get,

$$u_t(x,0)=u(x,0)-u_s(x)$$

$$=\frac{20}{l}x+20-\left(\frac{-40}{l}x+50\right)=\frac{60}{l}x-30------(16)$$

Now for the function $u_t(x,t)$ **we have the following boundary conditions,**

(i) $u_t(0,t)=0$

(ii) $u_t(l,t)=0$

(iii) $u_t(x,0)=\frac{60}{l}x-30$

Solving the equation for (1) for $u_t(x,t)$ by the method of separation of variables we get the solution of the form

$$u_t(x,t)=(A\cos px+B\sin px)e^{-\alpha^2p^2t}\;-----(17)$$

Substituting the condition (i) and (ii) in (17) we get,the most general solution of the form,

$$u_t(x,t)=\sum_{n=1}^{\infty}\mathrm{B_n}\sin\frac{\mathrm{n}\pi\mathrm{x}}{l}e^{-\frac{\alpha^2n^2\pi^2t}{l^2}}\qquad ....(18)$$

Apply the condition (c) in (18) we get

$$u_t(x,0)=\sum_{n=1}^{\infty}\mathrm{B_n}\sin\frac{\mathrm{n}\pi\mathrm{x}}{l}=\frac{60\mathrm{x}}{l}\text{-30}\qquad ....(19)$$

find B_n exp *and* $\frac{60\mathrm{x}}{l}$-30 in a half range fourier sine series

$$\frac{60\mathrm{x}}{l}\text{-30}=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}..........(20)$$

From (19) and (20) we get

$$B_n=b_n$$

To find B_n: The half range Fourier sine series $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

$$= \frac{2}{l}\int_0^l \left(\frac{60x}{l}-30\right)\sin\frac{n\pi x}{l}dx$$

$$= \frac{60}{l^2}\int_0^l (20x-l)\sin\frac{n\pi x}{l}dx$$

$$B_n = b_n = \frac{60}{l^2}\left[((2x-l))\left[-\cos\frac{\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]-(2)\left[-\sin\frac{\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^l$$

$$= \frac{60}{l^2}\left[-((2x-l))\left[\cos\frac{\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]+(2)\left[\sin\frac{\frac{n\pi x}{l}}{\left(\frac{n^2\pi^2}{l^2}\right)}\right]\right]_0^l$$

$$= \frac{60}{l^2}\left[\left\{-((2l-l))\left[\frac{\cos n\pi}{\frac{n\pi}{l}}\right]+(2)\left[\frac{\sin n\pi}{\frac{n^2\pi^2}{l^2}}\right]\right\}-\left\{-(0-l)\left(\frac{\cos 0}{\frac{n\pi}{l}}\right)+2\left(\frac{\sin 0}{\frac{n^2\pi^2}{l^2}}\right)\right\}\right]$$

$$= \frac{60}{l^2}\left[\frac{-l^2}{n\pi}\cos n\pi - \frac{l^2}{n\pi}\right]$$

$$= \frac{-60}{n\pi}\left[1+(-1)^n\right]$$

$$b_n = \begin{cases} 0, \ if \ \ n \ \ is \ \ odd \\ \frac{-120}{n\pi} \ \ if \ \ n \ \ is \ \ even \end{cases} \qquad ....(20)$$

Sub (20) in (18).we get

$$u_t(x,t) = \sum_{n=2,4,...}^{\infty} \frac{-120}{n\pi}\sin\frac{n\pi x}{l}e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

replace l by 10.*we get*

$$u_t(x,t) = \sum_{n=2,4,...}^{\infty} \frac{-120}{n\pi}\sin\frac{n\pi x}{10}e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$

$$u(x,t) = u_s(x) + u_t(x,t).$$

$$u(x,t) = -4x + 50 + \sum_{n=2,4,...}^{\infty} \frac{-120}{n\pi} \sin\frac{n\pi x}{10} e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} \quad .$$

7. STEADY STATE CONDITIONS AND ZERO BOUNDARY

26. A rod of length lcm long has its ends A and B kept at 0˚C and 100˚ C respectively until the steady state conditions prevail. If the temperature at B is suddenly reduced to 0˚C and maintained at 0˚C . Find the temperature distribution u(x,t) at a distance x from A at any time t. **[AU N/D 2017]**

Solution:

The temperature function u(x,t) is the solution of the one dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad --------(1)$$

When the steady state condition prevails $\frac{\partial u}{\partial t} = 0$ hence $\frac{\partial^2 u}{\partial x^2} = 0 \quad ---------(2)$

Hence , boundary conditions are $(i) \quad u(0) = 0, \quad (ii) \quad u(l) = 100$

The solution of (2) is u(x)=ax+b ---------(3)

Now applying condition (i) in (3),

$$u(0) = 0 + b$$

$$0 = b$$

$$u(x) = ax \quad -----(4)$$

Now applying condition (ii) in (4),

$$u(l) = al = 100$$

$$a = \frac{100}{l} \quad -----(5)$$

Substituting (5) in (4), $u(x) = \frac{100}{l}x$

Therefore in the steady state the temperature function is given by $u(x) = \frac{100}{l}x$. Now the end B is reduced to zero. At this stage the steady state is changed into unsteady state. For this unsteady state the initial temperature distribution is $u(x) = \frac{100}{l}x$.

The heat flow equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

The new boundary conditions are

(a) u(0,t)=0 for all t≥0

(b) u(l,t)=0 for all t≥0

(c) u(x,0)= $\frac{100x}{l}$

Now the suitable solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \quad -----(6)$$

Apply condition (c) in (6)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{100x}{l} \quad -----(7)$$

To find B_n: The half range Fourier sine series

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \frac{100x}{l} \quad ------(8)$$

From (7) and (8)

$$B_n = b_n$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = b_n = \frac{2}{l}\int_0^l \frac{100x}{l}\sin\frac{n\pi x}{l}dx$$

$$B_n = b_n = \frac{200}{l^2}\left[(x)\left[-\cos\frac{\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]-(1)\left[-\sin\frac{\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^l$$

$$= \frac{200}{l^2}\left[\frac{\left(-l^2\cos n\pi\right)}{n\pi}\right]$$

$$= \frac{200}{n\pi}\left[\left((-1)^{n+1}\right)\right] -----(9)$$

Now equation (6) become

$$u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\right]\sin\left(\frac{n\pi x}{l}\right)e^{\frac{-\alpha^2n^2\pi^2t}{l^2}}$$

RESULT: $$u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\right]\sin\left(\frac{n\pi x}{l}\right)e^{\frac{-\alpha^2n^2\pi^2t}{l^2}}$$

8. FINITE PLATES

27. Find the steady state temp. distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions u(0,y)= u(a,y)=0 for 0≤y≤b u(x,b)= 0 and u(x,0)=x(a-x) for 0≤x≤a **[AU N/D 2012**]

Solution:

The equation to be solved in $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

The boundary conditions are

(i)u(0,y)=0

(ii)u(a,y)=0

(iii)(x,b)=0

(iv)u(x,0)= x(a-x)

Suitable solution is

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py})\text{ ---------------------(1)}$$

Apply (i) in (1)

$$u(0,y)=A(Ce^{py}+De^{-py})=0$$

A=0 $\left[\because (Ce^{py}+De^{-py})\neq 0\right]$

Sub., A=0 in (1)

$$u(x,y)=B\sin px(Ce^{py}+De^{-py})\text{ -----------------------------(2)}$$

Apply (ii) in (2)

$$u(a,y)=B\sin pa(Ce^{py}+De^{-py})=0$$

Sinpa=0 $\left[\because (Ce^{py}+De^{-py})\neq 0 \text{ and } B\neq 0\right]$

Sinpa = sinnπ

Pa = nπ

$$p=\frac{n\pi}{a}$$

Now, (2) becomes

$$u(x,y) = B\sin\frac{n\pi}{a}x(Ce^{\frac{n\pi}{a}y} + De^{-\frac{n\pi}{a}y}) \text{ -------------------------------------(3)}$$

Apply (iii) in (3)

$$u(x,b) = B\sin\frac{n\pi}{a}x(Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}}) = 0$$

$$(Ce^{\frac{n\pi}{a}b} + De^{-\frac{n\pi}{a}b}) = 0 \qquad \left[\because \sin\frac{n\pi}{a}x \neq 0 \text{ and } B \neq 0\right]$$

$$De^{-\frac{n\pi}{a}b} = -Ce^{\frac{n\pi}{a}b}$$

$$D = -Ce^{\frac{2n\pi}{a}b}$$

Now, (3) becomes

$$u(x,y) = B\sin\frac{n\pi}{a}x(Ce^{\frac{n\pi}{a}y} - Ce^{\frac{2n\pi}{a}b}e^{-\frac{n\pi}{a}y})$$

$$u(x,y) = BC\sin\frac{n\pi}{a}x\left(e^{\frac{n\pi}{a}y} - \frac{e^{\frac{n\pi}{a}b}e^{-\frac{n\pi}{a}y}}{e^{\frac{-n\pi}{a}b}}\right) = BC\sin\frac{n\pi}{a}x\left(\frac{e^{\frac{n\pi}{a}y}e^{\frac{-n\pi}{a}b} - e^{\frac{n\pi}{a}b}e^{-\frac{n\pi}{a}y}}{e^{\frac{-n\pi}{a}b}}\right)$$

$$u(x,y) = -BCe^{\frac{n\pi}{a}b}\sin\frac{n\pi}{a}x\left(e^{\frac{n\pi}{a}(b-y)} - e^{\frac{-n\pi}{a}(b-y)}\right)$$

$$u(x,y) = -2BCe^{\frac{n\pi}{a}b}\sin\frac{n\pi}{a}x\sinh\frac{n\pi}{a}(b-y)\text{.........................(4)}$$

The most general form is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x\sinh(b-y)\frac{n\pi}{a}\text{ ....(5)}$$

Apply condition(iv),we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x\sinh b\frac{n\pi}{a} = x(a-x)\text{ ....(6)}$$

Using half range sine series ,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x = x(a-x), \quad where\ B_n = A_n \sinh b\frac{n\pi}{a}\text{ ....(7)}$$

From equation (6) and(7)

$$B_n = A_n \sinh b\frac{n\pi}{a}$$

Using half range sine series ,

To find B_n**: The half range Fourier sine series** $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

$$B_n = b_n = \frac{2}{30}\int_0^{30}(ax - x^2)\sin\frac{n\pi x}{a}dx$$

$$B_n = b_n = \frac{2}{30}\left[\left((ax-x^2)\right)\left[-\cos\frac{\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right] - (a-2x)\left[-\sin\frac{\frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^2}\right] + (-2)\left[\cos\frac{\frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^3}\right]\right]_0^a$$

$$B_n = \frac{2}{a}\left[-2\left(\frac{a}{n\pi}\right)^3(-1)^n + 2\left(\frac{a}{n\pi}\right)^3\right]$$

From equation (5)

$$A_n = \begin{cases} \frac{8a^2}{n3\pi 3} & if \ \ n = odd \\ 0 & if \ \ n = even \end{cases}$$

$$u(x,y) = \sum_{n=odd}^{\infty}\frac{8a^2}{n^3\pi^3\sin\frac{n\pi\pi}{a}}\ \sin\frac{n\pi}{a}x\sinh\frac{n\pi(b-y)}{a}$$

RESULT: $u(x,y) = \sum_{n=odd}^{\infty}\frac{8a^2}{n^3\pi^3\sin\frac{n\pi\pi}{a}}\ \sin\frac{n\pi}{a}x\sinh\frac{n\pi(b-y)}{a}$

9. ONE DIMENSIONAL HEAT FLOW

28. Solve $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$ subject to the conditions **[A.U. NOV 2013, MAY -2015]**

(i) $u(0,t) = 0 \ \ for \ \ all \ \ t \geq 0$

(ii) $u(l,t) = 0 \ \ for \ \ all \ \ t \geq 0$

(iii) $u(x,0) = \begin{cases} x & , 0 \leq x \leq l/2 \\ l - x, & l/2 \leq x \leq l \end{cases}$

Solution: **The temperature function u(x, t) satisfies the one dimensional heat equation is** $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$

Hence , boundary and initials conditions are

(i) $u(0,t)=0 \quad for \quad all \quad t \geq 0$

(ii) $u(l,t)=0 \quad for \quad all \quad t \geq 0$

(iii) $u(x,0)=\begin{cases} x & , 0 \leq x \leq l/2 \\ l-x, & l/2 \leq x \leq l \end{cases}$

Now the suitable solution is

$$u(x, t)= (A \cos px + B \sin px)\, e^{-\alpha^2 p^2 t} \quad \text{------------------------------(1)}$$

Apply condition (i) in (1)

$$u(0,t)= A\, e^{-\alpha^2 p^2 t} = 0$$

$$A=0 \quad [\because e^{-\alpha^2 p^2 t} \neq 0]$$

Now (1) implies $u(x, t)= B \sin px\, e^{-\alpha^2 p^2 t}$ --------------------------------------(2)

Apply condition (ii) in (2)

$$u(l, t)= B \sin l p\, e^{-\alpha^2 p^2 t} = 0$$

$$\sin pl = 0 \quad [\because e^{-\alpha^2 p^2 t} \neq 0 \text{ and } B \neq 0]$$

$$\sin pl = \sin n\pi$$

$$p = \frac{n\pi}{l}$$

Now (2) implies $u(x,t)= B \sin \frac{n\pi x}{l}\, e^{-\alpha^2 \left(\frac{n\pi}{l}\right)^2 t}$

The most general solution is

$$u(x, t)= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}\, e^{-\alpha^2 \left(\frac{n\pi}{l}\right)^2 t} \quad \text{--(3)}$$

Apply condition (iii) in (3)

$$u(x,0)= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = \begin{cases} x & , 0 \leq x \leq l/2 \\ l-x & , l/2 \leq x \leq l \end{cases} = f(x) \quad \text{……………..(4)}$$

Which is half range sine series in the interval (0,l)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{…………………………………………………………(5)}$$

From (4) and (5) ,we get

$B_n = b_n$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l/2}\right]+\frac{2}{l}\left[\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{l/2}^{l}\right]$$

$$B_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2\frac{t}{l^2}}$$

RESULT:

$$u(x,t) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2\frac{t}{l^2}}$$

Anna University Important Questions

PART-B

1. ZERO INITIAL VELOCITY (METHOD-1)

1. Tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$ apart. Motion is Started by displacing the string in to the from $y = k(lx - x^2)$ from which it is released at time t=0. Find the displacement of any point of the string at a distance of x from one end at time t. **Page No.11** **[AU - MAY-2015, N/D 17, A/M 18]**

2. A Tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At the time t=0 , the string is given by a shape defined by $f(x) = kx^2(l - x)$, where k is constant , and then released from rest . Find the displacement of any point 'x' of the string at any time t>0.
Page No.13 **[AU / MAY -2010,2008]**

3. A Tightly stretched string with fixed end point $x = 0$ and $x = l$ is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position .**Pg. No.15**
Find the displacement y at any distance x from one end at any time t. **[AU / DEC -2012]**

4. A string of length 2l is fastend at both ends. The mid point of the string is taken to a height b and then released from rest in that position. Show that the displacement is
$$y(x,t) = \frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-2)^2} \sin\left(\frac{(2n-1)\pi x}{2l}\right) \cos\left(\frac{(2n-1)\pi at}{2l}\right),$$ **Page No:18[[A.U A/M 2017]**

5. A taut string of length L has its ends $x = 0$ and $x = l$ fixed . The point where $x = \frac{l}{3}$ is drawn aside a small distance h , the displacement $y(x,t)$ satisfies $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$. Determine $y(x,t)$ at any time t. **Page No:21** [**AU / MAY -2010]**

6. A String is stretched and fastened to points at a distance l apart the motion is started by displace the string in form $y = a \sin \frac{\pi x}{l}, .0 < x < l$ from which it is released at a time t = 0 find the displacement at any time t. **Page No:25** **[AU / MAY -2014]**

2. NON ZERO VELOCITY (METHOD-2)

7. A tightly stretched string with fixed end points x = 0 and x = 1 is initially at rest in its equilibrium Position. It is set vibrating string giving each point a velocity $\lambda x(1 - x)$
Page No:27 **[AU / MAY -2013]**

8. A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v where $v = \begin{cases} \frac{cx}{l} in \ 0 < x < l \\ \frac{c(2l-x)}{l} in \ 0 < x < 2l \end{cases}$

x being the distance from one end point. Find the displacement of the string at any time. **Page No:30** **[AU / DEC -2010]**

9. A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity given by $v = \begin{cases} cx & if \ 0 \le x \le \frac{l}{2} \\ c(l-x) & if \ \frac{l}{2} \le x \le l \end{cases}$

Find the displacement function y (x,t). **Page No:33** **[AU M/J 2007, N/D 2010]**

10. If a string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}, \ 0 < x < l$, determine the transverse displacement (or) displacement of a point distant x from one end at time't'. **Page No:36**

[AU N/D 2008/NOV-2013,2014]

11. A Tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At the time t=0 , the string is given by a shape defined by $y(x,0) = \begin{cases} \frac{2kx}{l} & if \ 0 \le x \le \frac{l}{2} \\ 2k - \frac{2kx}{l} & if \ \frac{l}{2} \le x \le l \end{cases}$ where k is constant , and then released from rest . Find the displacement of any point 'x' of the string at any time t>0**.** **Page No:38** **[AU / NOV -2015]**

12. If a string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right),$, $0 < x < l$, determine the transverse displacement (or) displacement of a point distant x from one end at time't'.

Page No:41 **[AU N/D 2016**

3. HORIZONTALLY INFINITE PLATE

13. An infinitely long rectangular plate with insulated surface is 10cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x = 0$ is

kept at temperature given by $u = \begin{cases} 20\,y & , \quad for\ 0 \le y \le 5 \\ 20(10-y)\ , & for\ 5 \le y \le 10 \end{cases}$ find the steady state temperature distribution in the plate. **Page No:** 44 **[AU / MAY -2014]**

14. An infinitely long rectangular plate with insulated surface is 20cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x = 0$ is kept at temperature given by $u = \begin{cases} 10\,y & , \quad for\ 0 \le y \le 10 \\ 10(20-y)\ , & for\ 10 \le y \le 20 \end{cases}$ find the steady state temperature distribution in the plate **Page No:46** **[AU A/ M -2017]**

15. An infinitely long metal plate in the form of an area is enclosed between the lines $y = 0\ \ and\ \ y = \pi$ for $x > 0$. The temperature is zero along the edges $y = 0\ \ and\ \ y = \pi$ and at infinity. If the edges $x = 0$ is kept at a constant temperature $T_0{}^0C$,find the steady state temperature at any point of the plate?. **Page No:** 49 **[AU / DEC -2009]**

16. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angle to them. The breadth of this edge $x = 0\ is\ \ \pi$, this end is maintained at temperature as $u = k(\pi y - y^2)$ at all points while the other edges are at zero temperature. Determine the temperature u(x,y) at any point of the plate in the steady state if u satisfies Laplace equation? **. Page No: 51** **[AU / MAY -2010]**

17. An infinitely long plate in the form of an area is enclosed between the lines $y = 0\ \ and\ \ \ y = \pi$ for positive value of x. The temperature is zero along the edges $y = 0\ \ and\ \ \ y = \pi$ and the edge at infinity. If the edge $x = 0$ is kept at the temperature $f(y) = ky,\ 0 < y < \pi$ find the steady state temperature distribution in the plate?

Page No:54 **[AU / MAY -2010]**

4. VERTICALLY INFINITE PLATE

18. A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge y=0 is given by $u = \begin{cases} 20\ x\ \ for\ \ 0 \le x \le 5 \\ 20(10-x)\ \ for\ 5 \le x \le 10 \end{cases}$ and all the other three edges are kept at $0^\circ C$. Find the steady state temperature at any point in the plane. **Page No:** 56 **[AU / MAY -2013]**

19. A long rectangular plate has its surfaces insulated and the two long sides as well as one of the short sides are maintained at $0^\circ C$. Find an expression for the steady state temperature

$u(x, y)$ if the short side $y = 0$ *is* π *cm* long and is kept at $u_0^{\circ}C$. **Pg. No58** **[AU / MAY -2009]**

20. A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temperature along one short edge y=0 is given by $u(x,0) = 100\sin\frac{\pi x}{8}\ \ in\ 0 < x < 8$

While the two long edges x=0 and x=8 as well as the other short edges are kept at $0^{\circ}C$, find the steady state temperature function $u(x, y)$. **[AU / MAY -2010] pg. no60**

21. An infinitely long plane uniform plate is bounded by two parallel edges x=0 and x=l and an end at right angles to them. The breadth of this edge y=0 is l and is maintained at a temperature $f(x) = k(lx - x^2)$. All the other three edges are at $0^{\circ}C$. Find the steady state temperature at any interior point of the plate. **Page No:** 61 **[AU / MAY -2010]**

5.TWO DIMENSIONAL HEAT FLOW EQUATION (SQUARE PLATE)

22. A square plate is bounded by the lines x=0,y=0,x=20 and y=20. Its faces are insulated, the temperature along the edge y=20 is given by x (20 - x) while the other three edges are kept at $0^{\circ}C$.Find the steady-state temperature distribution on the plate. **Pg no**:64 **[AU / DEC,2011]**

23. A Square plate is bounded by the lines x= 0 , x=a, y = 0 and y = b. Its surfaces are insulated and the temperature along y=b is kept at 100^0 C. while the temperature along other three edges are at 0^0C. Find the steady-state temperature at any point in the plate.

Page No:63

[AU NOV-2014]

6. STEADY STATE CONDITIONS AND NON-ZERO BOUNDARY

24. The ends of A and B of a rod 10cm long have their temperature kept at 50˚ C and 100˚ C respectively until the steady state conditions prevail. The temperature of the end B is then suddenly reduced to 60˚ C and kept so while the end A is raised to 90˚ C .Find the temperature distribution function in the rod after time t. **Pg. No.68**. **[AU N/D 2008, 2015)]**

25. A bar 10cm long with insulated sides, has its ends A and B kept at 20˚ C and 40˚ C respectively until the steady state conditions prevail. The temperature of the end A is then suddenly reduced to 50˚ C and and at the same instant that at B is lowered to 10˚ C .Find the subsequent temperature at any point of the bar at any time.Pg. No; 71 **[AU A/M 2018]**

7. STEADY STATE CONDITIONS AND ZERO BOUNDARY

26. A rod of length lcm long has its ends A and B kept at 0˚C and 100˚ C respectively until the steady state conditions prevail. If the temperature at B is suddenly reduced to 0˚C and maintained at 0˚C . Find the temperature distribution u(x,t) at a distance x from A at any time t. Pg. No; 75 **[AU N/D 2017]**

8. FINITE PLATES

27. Find the steady state temp. distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions u(0,y)= u(a,y)=0 for $0 \le y \le b$ u(x,b)= 0 and u(x,0)=x(a-x) for $0 \le x \le a$.**Pg. No:77** **[AU N/D 2012]**

9. ONE DIMENSIONAL HEAT FLOW

28. Solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions **[A.U. NOV 2013, MAY -2015]**

(i) $u(0,t) = 0 \quad for \;\; all \;\; t \ge 0$

(ii) $u(l,t) = 0 \quad for \;\; all \;\; t \ge 0$

(iii) $u(x,0) = \begin{cases} x & , 0 \le x \le l/2 \\ l - x, & l/2 \le x \le l \end{cases}$ **Page No:79**

Anna University possible Questions

PART-A

APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

1. What are the possible solutions of one-dimensional wave equations?

[**AU –M/J 2006, Nov/Dec 2009/MAY-2014].**

2. In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ what does c^2? **[AU Nov/Dec 2011, June 2013]**

3. What is the basic difference between the solution of one dimensional wave equation and one dimensional heat equation? **[AU-May/June2012, N/D 17]**

4. Classify the Partial differential equation

$$(1 - x^2)z_{xx} - 2xyz_{xy} + (1 - y^2)z_{yy} + xz_x + 3x^2yz_y - 2z = 0$$

[AU – Nov-2014, May - 2015]

5. A rod 30 cm long has its ends A and B kept at $20°C$ and $80°C$ respectively until steady state conditions prevail. Find the steady state temperature in the rod.

[AU-May 2009, Apr 2008, May-2015]

6. State the laws assumed to derive the one dimensional heat equation. (OR) State the assumption in deriving the one dimensions heat flow equation (Unsteady State).

[AU- MAY /2014]

7. Given 3 possible solutions of the equation $\frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2}$ (Or) Write down the various possible solutions of one dimensional heat flow equation. **[AU Nov – 2014, A/M2018]**

8.Explain the term "steady state". **[A.U.NOV 2013]**

. 9. Write down the partial differential equation that represents steady state heat flow in two dimensional and name the variables involved. **[AU-May/June 2012]**

10. write down the p.d.e equation that represents steady state heat flow two dimensional and name the variables involved. **[A.U.M/J 2012]**

11.A rod 40 cm long with insulated sides with insulated sides has its ends A and B kep at $20^\bullet c$ and $60^\bullet c$ .Find the steady state temperature at a location 15 cm from A.

[A.U. A/M 2011]

12. write down the three possible solutions of Laplace equations in two-dimensions.

[A.U. N/D 2010, N/D 17]

13. Write down the boundary conditions for the following boundary value problem “If a string of length 'l' initially at rest in its equilibrium position and each of its point is given the velocity $\left(\frac{\partial y}{\partial t}\right)_t = 0 = v_0 \sin^3\left(\frac{\pi x}{l}\right) in\ 0 < x < l$

Determine the displacement function $y(x,t)$? **[A.U. A/M 2010]**

14. Classify the Partial differential equation $U_{XX} = U_X U_Y + XY$ **[A.U. A/M 2018]**

15. Classify the Partial differential equation **[AU – Nov-2008]**

$$3U_{xx} + 4U_{xy} + 6U_{yy} - U_x + U_y - U = 0$$

16. Classify the Partial differential equation. **[AU – A/M2008]**

$$3U_{xx} + 4U_{xy} - 2U_x + 3U_y = 0$$

17. A rod 50 cm long has its ends A and B kept at 20 and 70 degree respectively until steady state conditions prevail. Find the steady state temperature in the rod. **[AU-,N/D 2008]**

18. A rod 10 cm long has its ends A and B kept at 20 and 70 degree respectively until steady state conditions prevail. Find the steady state temperature in the rod. **[AU-,M/J 2008]**

19. In steady state conditions derive the solution of one dimensional heat flow equation

20. State one dimensional heat equation with the initial and final boundary conditions. **[AU-N/D 2006]**

The following boundary &initial conditions

21. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subject to initial displacement f(x) and initial velocity g(x). **[AU-N/D 2006, 2007]**

22. Classify the Partial differential equation

23. An insulated rod of length l=60 cm has its ends at A and B maintained at 30°C and 40°C respectively. Find the steady state solution . **[AU – Nov-2012]**

24. A rectangular plate is bounded by the linear line x=0,y=0,x=l and y=l .Its surface is insulated. The edge coinciding with x-axis is kept at $100^\circ c$. The edge coinciding with y-axis is kept at $50^\circ c$ .The other two edges are kept at $0^\circ c$. Write down the boundary conditions that are needed for solving two dimensional heat flow equation. **[AU Nov/Dec 2012,2011]**

25. Write down the two dimensional heat equation both in transient and Steady state. **[AU May/Jun 2012]**

26. Explain the initial and Boundary value problems. **[AU-Apr 2009]**

27. State the assumptions made in the derivation of one dimensional wave equation.
28. State Fourier law of conduction.
29. Distinguish between steady and unsteady states condition in one dimensional heat flow equation.

30. By the method of separation of variables solve $3x\frac{\partial u}{\partial x}-2y\frac{\partial u}{\partial y}=0$

31 **.** Classify the Partial differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$

[A.U. N/D 2009,2016]

33. By the method of separation of variables solve $\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u$

[**AU-M/J 2017]**

1. ZERO INITIAL VELOCITY (METHOD-1)

PART-B

1. Tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$ apart. Motion is Started by displacing the string in to the from $y=k(lx-x^2)$ from which it is released at time t=0. Find the displacement of any point of the string at a distance of x from one end at time t. **[AU -, MAY-2015, N/D 17, A/M 18**]

2. A Tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At the time t=0 , the string is given by a shape defined by $f(x)=kx^2(l-x)$, where k is constant , and then released from rest . Find the displacement of any point ‘x’ of the string at any time t>0.

[AU / MAY -2010,2008]

3. A Tightly stretched string with fixed end point $x=0$ and $x=l$ is initially in a position given by $y(x,0)=y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position.

Find the displacement y at any distance x from one end at any time t. **[AU / DEC -2012]**

4. A string of length 2l is fastend at both ends. The mid point of the string is taken to a height b and then released from rest in that position. Show that the displacement is

$$y(x,t)=\frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-2)^2}\sin\left(\frac{(2n-1)\pi x}{2l}\right)\cos\left(\frac{(2n-1)\pi at}{2l}\right)$$

5. A taut string of length L has its ends $x=0$ and $x=l$ fixed . The point where $x=\frac{l}{3}$ is drawn aside a small distance h , the displacement $y(x,t)$ satisfies $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$. Determine $y(x,t)$ at any time t. **[AU / MAY -2010]**

6. A String is stretched and fastened to points at a distance l apart the motion is started by displace the string in form $y=a\sin\frac{\pi x}{l},.0<x<l$ from which it is released at a time t = 0 find the displacement at any time t. **[AU / MAY -2014]**

2. NON ZERO VELOCITY (METHOD-2)

7. A tightly stretched string with fixed end points x = 0 and x = 1 is initially at rest in its equilibrium Position. It is set vibrating string giving each point a velocity $\lambda x(1-x)$

[AU / MAY -2013]

8. A string is stretched between two fixed points at a distance 2l apart and the points of the string are given initial velocities v where

$$v=\begin{cases}\frac{cx}{l} in \ 0<x<l\\ \frac{c(2l-x)}{l} in \ 0<x<2l\end{cases}$$

x being the distance from one end point. Find the displacement of the string at any time. **[AU / DEC -2010]**

9. A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity given by

$$v=\begin{cases}cx & if\ 0\le x\le\frac{l}{2}\\ c(l-x) & if\ \frac{l}{2}\le x\le l\end{cases}$$

Find the displacement function y (x,t). **[AU M/J 2007, N/D 2010]**

10. If a string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0}=v_0\sin^3\frac{\pi x}{l}$, $0<x<l$, determine the transverse displacement (or) displacement of a point distant x from one end at time't'.

[AU N/D 2008/NOV-2013,2014]

3. HORIZONTALLY INFINITE PLATE

11. An infinitely long rectangular plate with insulated surface is 10cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x = 0$ is kept at temperature given by $u = \begin{cases} 20y & , \ for \ 0 \le y \le 5 \\ 20(10-y), & for \ 5 \le y \le 10 \end{cases}$ find the steady state temperature distribution in the plate **[AU / MAY -2014]**

12. An infinitely long metal plate in the form of an area is enclosed between the lines $y = 0 \ and \ y = \pi$ for $x > 0$. The temperature is zero along the edges $y = 0 \ and \ y = \pi$ and at infinity. If the edges $x = 0$ is kept at a constant temperature $T_0{}^0C$,find the steady state temperature at any point of the plate? **[AU / DEC -2009]**

13. An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angle to them. The breadth of this edge $x = 0 \ is \ \pi$, this end is maintained at temperature as $u = k(\pi y - y^2)$ at all points while the other edges are at zero temperature. Determine the temperature u(x,y) at any point of the plate in the steady state if u satisfies Laplace equation? **[AU / MAY -2010]**

14. An infinitely long plate in the form of an area is enclosed between the lines $y = 0 \ and \ y = \pi$ for positive value of x. The temperature is zero along the edges $y = 0 \ and \ y = \pi$ and the edge at infinity. If the edge $x = 0$ is kept at the temperature $f(y) = ky, \ 0 < y < \pi$ find the steady state temperature distribution in the plate?

[AU / MAY -2010]

4. VERTICALLY INFINITE PLATE

15. A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge y=0 is given by

$$u = \begin{cases} 20\ x \ \ for \ \ 0 \le x \le 5 \\ 20(10-x) \ \ for \ 5 \le x \le 10 \end{cases}$$

and all the other three edges are kept at $0^\circ C$. Find the steady state temperature at any point in the plane. **[AU / MAY -2013]**

16. A long rectangular plate has its surfaces insulated and the two long sides as well as one of the short sides are maintained at $0^\circ C$. Find an expression for the steady state temperature $u(x, y)$ if the short side $y = 0 \ is \ \pi \ cm$ long and is kept at $u_0^\circ C$. **[AU / MAY -2009]**

17. A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in the length without introducing an appreciable error. If the temperature along one short edge y=0 is given by

$$u(x,0) = 100\sin\frac{\pi x}{8} \ \ in\ 0 < x < 8$$

While the two long edges x=0 and x=8 as well as the other short edges are kept at $0^\circ C$, find the steady state temperature function $u(x, y)$ **[AU / MAY -2010]**

18. An infinitely long plane uniform plate is bounded by two parallel edges x=0 and x=l and an end at right angles to them. The breadth of this edge y=0 is l and is maintained at a temperature

$f(x) = k(lx - x^2)$. All the other three edges are at $0^\circ C$. Find the steady state temperature at any interior point of the plate. **[AU / MAY -2010]**

5.TWO DIMENSIONAL HEAT FLOW EQUATION (SQUARE PLATE)

19. A square plate is bounded by the lines x=0,y=0,x=20 and y=20. Its faces are insulated, the temperature along the edge y=20 is given by x (20 - x) while the other three edges are kept at $0^\circ C$. Find the steady-state temperature distribution on the plate. **[AU / DEC -2009,2011]**

20. A Square plate is bounded by the lines x= 0 , x=a, y = 0 and y = b. Its surfaces are insulated and the temperature along y=b is kept at 100^0 C. while the temperature along other three edges are at 0^0C. Find the steady-state temperature at any point in the plate.

[AU NOV-2014]

6.STEADY STATE CONDITIONS AND NON-ZERO BOUNDARY

21. The ends of A and B of a rod 10cm long have their temperature kept at 50˚ C and 100˚ C respectively until the steady state conditions prevail. The temperature of the end B is then suddenly reduced to 60˚ C and kept so while the end A is raised to 90˚ C .Find the temperature distribution function in the rod after time t. **[AU N/D 2008, 2015)]**

22. A bar 10cm long with insulated sides, has its ends A and B kept at 20˚ C and 40˚ C respectively until the steady state conditions prevail. The temperature of the end A is then suddenly reduced to 50˚ C and and at the same instant that at B is lowered to 10˚ C .Find the subsequent temperature at any point of the bar at any time. **[AU A/M 2018]**

7. STEADY STATE CONDITIONS AND ZERO BOUNDARY

23. A rod of length lcm long has its ends A and B kept at 0˚C and 100˚ C respectively until the steady state conditions prevail. If the temperature at B is suddenly reduced to 0˚C and maintained at 0˚C . Find the temperature distribution u(x,t) at a distance x from A at any time t **[AU N/D 2017]**

8. FINITE PLATES

24. Find the steady state temp. distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions u(0,y)= u(a, y)=0 for 0≤y≤b u(x, b)= 0 and u(x,0)=x(a-x) for 0≤x≤a **[AU N/D 2012]**

9. ONE DIMENSIONAL HEAT FLOW

26. Solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions **[A.U. NOV 2013, MAY -2015]**

(i) $u(0,t) = 0 \quad for \ \ all \ \ t \geq 0$

(ii) $u(l,t) = 0 \quad for \ \ all \ \ t \geq 0$

(iii) $u(x,0) = \begin{cases} x \quad , 0 \leq x \leq l/2 \\ l - x, \ l/2 \leq x \leq l \end{cases}$

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II Year B.E (Civil, EEE & Mech.)

DEPARTMENT OF MATHEMATICS

SUBJECT NAME : MA8353 - TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS

Unit -IV (FOURIER TRANSFORM)

Syllabus: Statement of Fourier integral theorem – Fourier transforms pair – Fourier sine and cosine transforms- Properties- Transforms of simple functions- Convolution theorem – Parseval's identity.

No. of pages : 71+1 **Cost per set:** Rs.

Updated Questions:

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Part - B	2	23	4	26
	32	59	7	32
	33	61	21	47

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PRINCIPAL

UNIT – IV
FOURIER TRANSFORMS
PART – A

1. State Fourier integral theorem

[A.U. April 1996, April/May 2005, May-2016]

Solution:

If f(x) is piece wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(x-t)} dt ds$$

or equivalently

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x) dt d\lambda$$

This is known as Fourier integral theorem or Fourier integral formula.

2. Show that $f(x) = 1, 0 < x < \infty$ cannot be represented by a Fourier integral.

[A.U. April/May 2003]

Solution:

$$\int_0^{\infty} |f(x)| dx = \int_0^{\infty} 1 dx = [x]_0^{\infty} = \infty$$

and this value tends to ∞ as $x \to \infty$

ie., $\int_0^{\infty} f(x) dx$ is not convergent.

Hence f(x) = 1 cannot be represented by a Fourier integral.

3. Define Fourier Transform pair. **[A U, March, 1996]**

Solution:

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \text{ and}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

4. What is the Fourier cosine transform of a function. (or)
Write down the Fourier cosine transform pair of formulae.

Solution: **[A U, October / November 1996]**

The infinite Fourier cosine transform f(x) is defined as

$$F_c f(x) = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx$$

The inverse Fourier cosine transform is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c\left[f(x)\right] \cos sx ds$$

5. Find the Fourier cosine transform of $e^{-ax}, a > 0$

Solution: **[A U, April, 2001]**

$$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$$

$$F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2}, a > 0$$

6. Find the Fourier cosine transform of e^{-3x}

Solution:

$$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$$

$$F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$F_c\left[e^{-3x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-3x} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{3}{s^2 + 3^2}\right]$$

7. Find the Fourier sine transform of e^{-3x}

Solution: **[A U, Nov / Dec 1996, M/J 2013]**

$$F_s\left[e^{-3x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-3x} \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2 + 3^2}\right]$$

Formula $F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sxdx$

8. Find the Fourier Sine transform of $\frac{1}{x}$ [AU A/M 2015, N/D 2016, A/M 2017]

Solution:

We know that

$$F_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sxdx$$

$$F_s\left[\frac{1}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sxdx$$

Let $sx=\theta$ $\quad x\to 0\Rightarrow\theta\to 0$

$sdx=d\theta$ $\quad x\to\infty\Rightarrow\theta\to\infty$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\left(\frac{s}{\theta}\right)\sin\theta\frac{d\theta}{s}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\sin\theta}{\theta}d\theta$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}\right]=\sqrt{\frac{\pi}{2}} \qquad \left[\because\int_0^\infty\frac{\sin\theta}{\theta}d\theta=\frac{\pi}{2}\right]$$

9. Define Fourier sine transform and its inversion formula.

Solution: [A.U. April/May, 2004]

The infinite Fourier sine transform of f(x) is defined as

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sxdx$$

The inversion formula is

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sxds$$

10. Find the Fourier sine transform of f(x) = e^{-x} and hence deduce that

$$\int_0^\infty\frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-m}$$ **[A U, March, 1998, 1999 & 2000]**

Solution:

$$F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sxdx=\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\ a>0$$

By the inversion formula for sine transform

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sxds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+1}\right)\sin sxds$$

$$\int_0^\infty\frac{x\sin mx}{1+s^2}dx=\frac{\pi}{2}f(x)=\frac{\pi}{2}e^{-x},$$

$$\Rightarrow\int_0^\infty\frac{x\sin mx}{1+s^2}dx=\frac{\pi}{2}e^{-m}$$

11. Find $F_c\left[xe^{-ax}\right]$ **and** $F_s\left[xe^{-ax}\right]$

Solution:

(i) $F_c\left[xf(x)\right]=\frac{d}{ds}F_s\left[f(x)\right]$

$$F_c\left[xe^{-ax}\right]=\frac{d}{ds}F_s\left[e^{-ax}\right]$$

$$=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sxdx\right]$$

$$=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)a^2}\right]$$

(ii) $F_s\left[xe^{-ax}\right]=-\frac{d}{ds}F_c\left[e^{-ax}\right]$

$$=-\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sxdx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$$

12. Show that the Fourier sine transforms of $xe^{-x^2/2}$ is self reciprocal.
[A U, March, 1996]

Solution:

We know that $Fs[xf(x)] = \frac{d}{ds}Fc[f(x)]$

Here $f(x) = e^{\frac{-x^2}{2}}$

$$\therefore \qquad F_C[f(x)] = Fc\left[e^{\frac{-x^2}{2}}\right] = \frac{e^{\frac{-s^2}{4a^2}}}{a\sqrt{2}}$$

$$F_s[xf(x)] = \frac{-d}{ds}\left[\frac{e^{\frac{-s^2}{4a^2}}}{a\sqrt{2}}\right]$$

i.e., $$F_s[xf(x)] = \frac{s}{2\sqrt{2}a^3}e^{\frac{-s^2}{4a^2}}$$

Deduction : put $a = \frac{1}{\sqrt{2}}$. Then we have $Fs\left[xe^{\frac{-x^2}{2}}\right] = se^{\frac{-s^2}{2}}$

$\Rightarrow xe^{\frac{-x^2}{2}}$ is self reciprocal w.r.t Fourier sine transform.

13. If Fourier transform of f(x) is F(s), prove that the Fourier transform of f(x) $\cos ax$ is $\frac{1}{2}[F(s-a)+F(s+a)]$ [AU April, 2001, Nov/Dec 2014]

(or)

State and Prove Modulation Theorem.

Proof:

If *F(s)* is the F.T of *f(x)*, then

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)]$$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \left(\frac{e^{iax} + a^{-iax}}{2} \right) dx$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx \right]$$

$$= \frac{1}{2} [F(s+a) + F(s-a)]$$

14. Prove that $F_C[f(x)\cos ax] = \frac{1}{2}[F_C(s+a) + F_C(s-a)]$ **where** $\mathbf{F_c}$ **denotes the Fourier cosine transform f(x). [A U April/May 2001]**

Solution:

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ax \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \left[\frac{\cos(s+a)x + \cos(s-a)x}{2} \right] dx$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(s+a)x dx + \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(s-a)x dx$$

$$= \frac{1}{2} [Fc(s+a) + Fc(s-a)]$$

15. If F(s) is the complex Fourier transform of f(x) then find $F[\delta(x-a)]$

Solution: [A U, April, 2000]

$$F[\delta(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \int_a^{a+h} \frac{1}{h} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{h \to 0} \frac{1}{h} e^{isx} \left[\frac{e^{ish} - 1}{ish} \right]$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \because \lim_{h \to 0} \frac{e^{\theta} - 1}{\theta} = 1$$

16. Given that $e^{-x^2/2}$ is self reciprocal under Fourier cosine transform, find (i) Fourier sine transform of $xe^{-x^2/2}$ and (ii) Fourier cosine transform of $x^2e^{-x^2/2}$ **[A U, DEC 1996]**

Solution:

Given $Fc\left[e^{-x^2/2}\right] = e^{-s^2/2}$

$$F_s\left[xe^{-x^2/2}\right] = -\frac{d}{ds}F_c\left[xe^{-x^2/2}\right]$$

$$= -\frac{d}{ds}\left[e^{-s^2/2}\right] = -e^{-s^2/2}\left[-s\right] = se^{-s^2/2}$$

$$F_C\left[x^2e^{-x^2/2}\right] = \frac{d}{ds}F_s\left[xe^{-x^2/2}\right]$$

$$= \frac{d}{ds}\left[se^{-s^2/2}\right] = \left[se^{-s^2/2}(-s) + e^{-s^2/2}\right]$$

$$= -s^2e^{-s^2/2} + e^{-s^2/2} = \left(1 - s^2\right)e^{-s^2/2}$$

17. If $F_c(s)$ is the Fourier cosine transform of *f(x)*, Prove that the Fourier cosine transform of *f(ax)* is $\frac{1}{a}F_c\left[\frac{s}{a}\right]$

Solution:

i.e., To prove:

$$F_c\left[f(ax)\right] = \frac{1}{a}F_c\left[\frac{s}{a}\right]$$

W.K.T $F_c\left[f(ax)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos sx\,dx$

Put ax = u when $x \to 0 \Rightarrow u \to 0, x \to \infty \Rightarrow u \to \infty$

adx = du

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\left(\frac{su}{a}\right)\frac{du}{a}$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\frac{s}{a}u\,du$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\frac{s}{a}t\,dt$$

$$= \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

18. If F(s) is the Fourier transform of *f(x)*, then find the Fourier transform of *f(x-a)*. (or) State and prove shifting theorem.
(or) If the Fourier Transform of f(x) is F[f(x)] = F(s), then show that F[f(x-a)] =e^{ias}F(S)

Solution: **[AU N/D 2013, A/M 2015, A/M 2017]**

Given: $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

put $t = x - a$ as $x \to -\infty \Rightarrow t \to -\infty$

dt = dx $x \to \infty \Rightarrow t \to \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(a+t)} dt$$

$$= \frac{e^{ias}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= e^{ias} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt \right] = e^{ias} F(s)$$

19. If $F_s(s)$ is the Fourier sine transform of *f(x)*,

show that $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

Solution:

$$F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos ax \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \cos ax dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} f(x) \frac{1}{2} [\sin(s+a)x + \sin(s-a)x] \right] dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\int_0^{\infty} f(x) \sin(s+a)x dx + \int_0^{\infty} f(x) \sin(s-a)x dx \right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s-a)x\,dx\right]$$

$$= \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$$

20. State the convolution theorem for Fourier transforms.

(or)

State the Faltung theorem. [A.U. April / May 2003, N/D 2017, A/M 2018]

If F(s) and G(s) are the Fourier transform of f(x) and g(x) respectively. Then the Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transform.

$$F[f(x)\,g(x)] = F(S)\,G(S) = F[f(x)]F[g(x)]$$

21. Find the Fourier transform of $f(x) = \begin{cases} 1 & in\ |x| < a \\ 0 & in\ |x| > a \end{cases}$ **[AU A/M 2018]**

Solution:

We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is}\right] = \frac{1}{\sqrt{2\pi}}\frac{2}{is}\left[\frac{e^{isa} - e^{-ias}}{2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{2}{s}\left[\frac{e^{isa} - e^{-ias}}{2i}\right]$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}\right]$$

22. Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x & if\ 0 < x < a \\ 0 & if\ \ x \geq a \end{cases}$

Solution:

$$F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos x \cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^a \cos x \cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_0^a \left[\cos(s+1)x + \cos(s-1)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1} \right]$$

Provided $s \neq 1; s \neq -1$

23. Find the Fourier Cosine transform of $e^{-2x} + 3e^{-x}$

Solution:

We know that

$$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$$

$$F_c\left[e^{-2x} + 3e^{-x}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[e^{-2x} + 3e^{-x}\right] \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cos sx dx + \sqrt{\frac{2}{\pi}} 3 \int_0^\infty e^{-x} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-2x}}{4+s^2} \left(-2\cos sx + s \sin sx\right) \right]_0^\infty$$

$$+ 3\sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} \left(-1\cos sx + s \sin sx\right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{4+s^2}(-2) \right] + 3\sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{1+s^2}(-1) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2}{4+s^2} \right] + 3\sqrt{\frac{2}{\pi}} \left[\frac{1}{1+s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2}{4+s^2} + \frac{3}{s^2+1} \right]$$

24. Find Fourier Cosine transform of e^{-x}**.**

Solution:

$$F_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sxdx$$

$$F_c\left[e^{-x}\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sxdx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1}{1+s^2}\right] \qquad \left[\because \int_0^{\infty} e^{-ax}\cos bxdx=\frac{a}{a^2+b^2}\right]$$

25. If Fourier transform of $f(x)=F(s)$ then what is Fourier transform of $f(ax)$ (OR) If F(s) is the fourier transforms of f(x), Prove that $F\left[f(ax)\right]=\frac{1}{a}F\left(\frac{s}{a}\right)$ (or) State and Prove a Change of scale property.

[AU M/J 2013, N/D 2015, N/D 2016, N/D 2017]

Proof:

For any non zero real a, $F\left[f(ax)\right]=\frac{1}{|a|}F\left(\frac{s}{a}\right)$

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[f(ax)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

$\left[\because put\ t=ax,\ dt=adx\right]$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{is\left(\frac{t}{a}\right)}\frac{dt}{a}$$

$$=\frac{1}{a}F\left[\frac{s}{a}\right]$$

26. If F denotes the Fourier transform operator then show that

$$F\left\{x^n f(x)\right\}=(-i)^n\frac{d^n}{ds^n}F\left\{s\right\}$$

Solution:

To Prove $F\left\{x^n f(x)\right\}=(-i)^n\frac{d^n}{ds^n}F\left\{s\right\}$

We have $F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

Differentiating both sides n times w.r.t 's' we get

$$\frac{d^n}{ds^n}F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)(ix)^n e^{isx}dx$$

$$=\frac{i^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)x^n e^{isx}dx=i^n F\left[x^n f(x)\right]$$

$$F\left[x^n f(x)\right]=\frac{1}{i^n}\frac{d^n}{ds^n}F(s)$$

Hence

$$=(-i)^n\frac{d^n}{ds^n}F(s)$$

27. If F(s) is the Fourier transform of $f(x)$, show that the Fourier transform of $e^{iax}f(x)$ is $F(s+a)$. **[AU Nov/Dec 2014]**

Solution:

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{iax}f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x}f(x)dx$$

$$=F(s+a)$$

28. Find $F\left[\frac{d^n}{dx^n}f(x)\right]$

Solution:

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[f^1(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^1(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}d\left[f(x)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) e^{isx} (is) dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ (0-0) - is \int_{-\infty}^{\infty} f(x) e^{isx} dx \right\}$$

$$= (-is) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \qquad [\because f(x) \to 0 \, as \ x \to \pm\infty]$$

$$= -isF(s)$$

Similarly

$$F\left[f^{(n)}(x) \right] = (-is)^n F(s) \quad if \quad f, f^1, f^{11}, \ldots\ldots.. f^{n-1} \to 0 \ as \, x \to \pm\infty$$

29. State Parseval’s Identity for Fourier transform.

Solution:

If F(s) is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

30. Find the Fourier transform of $f(x) = \begin{cases} e^{ikx} & , \ a < x < b \\ 0 \ x < a \ and \ x > b \end{cases}$

Solution:

We know that $F\left[f(x) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_a^b$$

$$= \frac{1}{i(k+s)\sqrt{2\pi}} \left[e^{i(k+s)b} - e^{i(k+s)a} \right]$$

$$= \frac{-i}{(k+s)\sqrt{2\pi}} \left[e^{i(k+s)b} - e^{i(k+s)a} \right]$$

$$= \frac{i}{(k+s)\sqrt{2\pi}}\left[e^{i(k+s)a} - e^{i(k+s)b}\right]$$

31. State the Fourier transform of the derivatives of a function. (OR) Find the Fourier transform of a derivative of the function f(x) if f(x) → 0 as x → ±∞
[AU N/D 2005, M/J -2016]

Solution: $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}d\{f(x)\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left\{e^{isx}f(x)\right\}_{-\infty}^{\infty} is\int_{-\infty}^{\infty} f(x)e^{isx}dx\right]$$

= - is F(s) if **f(x) → 0 as x → ±∞**

where $F(s)$ is the Fourier transform of $f(x)$.

Therefore $F[f'(x)] = (-is)F(s)$

32. Find the Fourier sine transform of e^{-x}.

Solution: **[AU M /J 2006]**

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx dx$$

$$F_s\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\sin sx dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+1^2}\right]$$

33. Find the Fourier Cosine transform of of $f(x)$ defined as **[AU N /D 2006]**

$$f(x) = \begin{cases} x \ \ for\ 0 < x < 1 \\ 2-x \ \ for\ 1 < x < 2 \\ 0 \ \ for \ \ x > 2 \end{cases}$$

Solution:

We know that

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx dx + \int_1^2 (2-x)\cos sx dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[x\frac{\sin sx}{s}-(1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1+\left[(2-x)\frac{\sin sx}{s}-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[x\frac{\sin sx}{s}+\left(\frac{\cos sx}{s^2}\right)\right]_0^1+\left[(2-x)\frac{\sin sx}{s}-\left(\frac{\cos sx}{s^2}\right)\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\left(\frac{\sin s}{s}+\frac{\cos s}{s^2}\right)-\left(\frac{1}{s^2}\right)\right]+\left[\left(0-\frac{\cos 2s}{s^2}\right)-\left(\frac{\sin s}{s}-\frac{\cos s}{s^2}\right)\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\frac{\sin s}{s}+\frac{\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}-\frac{\sin s}{s}+\frac{\cos s}{s^2}\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\frac{2\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}\right\}$$

$$=\frac{\sqrt{\frac{2}{\pi}}}{s^2}\{1-2\cos s-\cos 2s\}$$

34. Find the Fourier sine transform of $f(x)=e^{-ax}, a>0$ **[AU N/D 2007]**

Solution:

$$F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right),\ a>0$$

35. State inverse theorem for complex Fourier transform.

Solution:

Let $f(x)$ be a function satisfying Dirichlet's conditions in every finite interval $(-l,l)$.

Let $F(s)$ denote the Fourier transform of $f(x)$. Then at every point of continuity of $f(x)$, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

36. Find the function $f(x)$ **whose sine transform is** e^{-as}**. [AU M/J 2010]**

Solution:

Given $F_S[f(x)]=e^{-as}$

By inversion formula

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sxds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\sin sxds \qquad \left[\because \int_0^\infty e^{-ax}\sin bxdx=\frac{b}{a^2+b^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{x}{a^2+x^2}\right]$$

37. Find the Fourier cosine transform of e^{-2x}

Solution: **[AU N/D 2010]**

$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos xdx$$

$$F_c\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos xdx \qquad \left[\because \int_0^\infty e^{-ax}\cos bxdx=\frac{a}{a^2+b^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{2}{s^2+2^2}$$

38. Prove that $F_S[xf(x)]=-\frac{d}{ds}[F_C(s)]$ **[AU A/M 2011]**

Solution:

We know that $F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sxdx$

Differentiating both sides w.r.to s

$$\frac{d}{ds}F_c[f(x)]=\sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_0^\infty f(x)\cos sxdx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\frac{d}{ds}(\cos sx)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)(-x\sin sx)dx$$

$$=-\sqrt{\frac{2}{\pi}}\int_0^\infty xf(x)\sin sxdx$$

$$=-F_S[xf(x)]$$

$$(i.e.,)\qquad F_S[xf(x)]=-\frac{d}{ds}[F_C(s)]$$

39. Define the Fourier sine and Cosine transform of $f(x)$**.**

Solution:

The infinite Fourier Cosine transform of $f(x)$ is defined as

$$F_C\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx dx$$

The infinite Fourier Sine transform of $f(x)$ is defined as

$$F_S\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx dx$$

40. Find the Fourier transform of $e^{-\alpha|x|}, \alpha > 0$. **[AU N/D 2012]**

Solution:

We know that

$$F\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{-\alpha|x|}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{isx}dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{\alpha x}\cos sx dx=\sqrt{\frac{2}{\pi}}\left[\frac{\alpha}{s^2+\alpha^2}\right]$$

41. Find the Fourier sine transform of $\dfrac{e^{-ax}-e^{-bx}}{x}$

Solution:

$$F_S\left[\frac{e^{-ax}-e^{-bx}}{x}\right]=F_S\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$$

$$=F_S\left[\frac{e^{-ax}}{x}\right]-F_S\left[\frac{e^{-bx}}{x}\right]$$

$$\left[\because F_S\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)-\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$$

$$F_S\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right)-\tan^{-1}\left(\frac{s}{b}\right)\right]$$

42. Find the Fourier sine transform of $\dfrac{x}{x^2+a^2}$

Solution:

We know that F.S.T of e^{-ax} is given by

$$F_S\left[f(x)\right]=F_S\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

Using inversion formula for F.S.T we get

$$e^{-ax}=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\sin sxds$$

$$\int_0^\infty\frac{s}{s^2+a^2}\sin sxds=\frac{\pi}{2}e^{-ax},\ a>0$$

Changing x and s , we get

$$\int_0^\infty\frac{x}{x^2+a^2}\sin sxdx=\frac{\pi}{2}e^{-as},\ a>0$$

Now $$F_S\left[\frac{x}{x^2+a^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{x}{x^2+a^2}\sin sxdx$$

$$=\sqrt{\frac{2}{\pi}}\frac{\pi}{2}e^{-as}$$

$$F_S\left[\frac{x}{x^2+a^2}\right]=\sqrt{\frac{\pi}{2}}e^{-as}$$

43. Find the Fourier Cosine transform of $5e^{-2x}+2e^{-5x}$

Solution:

We know that

$$F_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sxdx$$

$$F_c\left[5e^{-2x}+2e^{-5x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\left[5e^{-2x}+2e^{-5x}\right]\cos sxdx$$

$$=5\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos sxdx+\sqrt{\frac{2}{\pi}}2\int_0^\infty e^{-5x}\cos sxdx$$

$$=5\sqrt{\frac{2}{\pi}}\left[\frac{e^{-2x}}{4+s^2}\left(-2\cos sx+s\sin sx\right)\right]_0^\infty$$

$$+2\sqrt{\frac{2}{\pi}}\left[\frac{e^{-5x}}{5^2+s^2}\left(-5\cos sx+s\sin sx\right)\right]_0^\infty$$

$$=5\sqrt{\frac{2}{\pi}}\left[0-\frac{1}{4+s^2}(-2)\right]+2\sqrt{\frac{2}{\pi}}\left[0-\frac{1}{25+s^2}(-5)\right]$$

$$=10\sqrt{\frac{2}{\pi}}\left[\frac{1}{4+s^2}+\frac{1}{s^2+25}\right]$$

44. If $f(x)=\begin{cases}\sin x & 0\le x\le a\\ 0 & x>a\end{cases}$ **, Find the Sine transform.**

Solution:

The Fourier Sine transform of $f(x)$ is given by

$$F_S[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sxdx$$

$$F_S[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^a \sin x\sin sxdx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^a \frac{1}{2}\{\cos(1-s)x-\cos(1+s)x\}dx$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin(1-s)x}{1-s}-\frac{\sin(1+s)x}{1+s}\right]_0^a$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(1-s)a}{1-s}-\frac{\sin(1+s)a}{1+s}\right]$$

45. Find the function $f(x)$ **whose sine transform is** $\frac{e^{-as}}{s}$**.**

Solution:

We know that the inverse Fourier sine transforms of $F_S[f(x)]$ is given by

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sxds$$

Here $F_S[f(x)]=\frac{e^{-as}}{s}$

$$\therefore\quad f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sxds$$

Differentiating w.r.t ‘x’ on both sides, we get

$$\frac{d}{dx}[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\frac{d}{dx}\sin sxds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\cos sx.sds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\cos sxds$$

$$=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}$$

ie., $\frac{d}{dx}[f(x)]=\sqrt{\frac{2}{\pi}}\frac{a}{x^2+a^2}$

Integrating,

$$f(x)=a\sqrt{\frac{2}{\pi}}\int\frac{1}{x^2+a^2}dx$$

$$= a\sqrt{\frac{2}{\pi}}\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$$

46. Define self reciprocal with respect to Fourier Transform. [AU N/D 2013]

Solution.

If a transformation of a function f(x) is equal to f(s) then the function f(x) is called self reciprocal.

47. State Convolution theorem. [AU N/D 2012]

Solution.

The Fourier transform of the Convolution of f(x) and g(x) is equal to the product of their Fourier transforms.

(i.e.,) $F[f(x)*g(x)] = F[f(x)]F[g(x)]$

48. Evaluate $\int_0^\infty \frac{s^2 ds}{(s^2+a^2)(s^2+b^2)}$ **using Fourier transforms [A.U N/D 2015]**

Solution

By Parseval's Identity $\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s[f(x)]F_s[g(x)]ds$ ………….(1)

$$f(x) = e^{-ax} \quad g(x) = e^{-bx}$$

We know that $F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$ $F_s[e^{-bx}] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+b^2}\right]$

L.H.S $\int_0^\infty f(x)g(x)dx = \int_0^\infty e^{-(a+b)x}dx \quad = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty \qquad = \left(0 + \frac{1}{(a+b)}\right)$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s[f(x)]F_s[g(x)]ds$$

$$\frac{1}{(a+b)} = \int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+b^2}\right]ds$$

$$\frac{1}{(a+b)} = \frac{2}{\pi}\int_0^\infty \left[\frac{s}{s^2+a^2}\right]\left[\frac{s}{s^2+b^2}\right]ds$$

$$\int_0^\infty \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]ds = \frac{\pi}{2(a+b)}$$

PART - B

PROBLEMS BASED ON FOURIER TRANSFORMS AND ITS INVERSION, PARSEVALS IDENTITY

FORMULA **(1) Fourier transform** $F[s]=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

(2) Inverse Fourier transform $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$

(3) Parseval's identity $\int_{-\infty}^{\infty} |f(x)|^2\, dx=\int_{-\infty}^{\infty} |F(s)|^2\, ds$

1. Find the Fourier Transform of $f(x)=\begin{cases} a-|x|, & for\ |x|< a \\ 0, & for\ |x|>a>0 \end{cases}$ **hence deduce that** $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$ **,** $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$ **[AU Nov / Dec, 1996]**

Solution:

Formula $F[s]=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

Given: Here $f(x)=\begin{cases} a-|x| & for\ |x|<a \quad -a<x<a \\ 0, & otherwise \end{cases}$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}[a-|x|]\ e^{isx}\ dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}[a-|x|]\,[\cos sx+i\sin sx]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a-x)\cos sxdx+0\right]$$ Since $[a-|x|]\cos sx$ is an even function.

$$=\sqrt{\frac{2}{\pi}}\int_0^a [a-x]\cos sxdx$$ $[a-|x|]\sin sx$ is an odd function.

$$=\sqrt{\frac{2}{\pi}}\left[(a-x)\frac{\sin sx}{s}-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[(a-x)\frac{\sin sx}{s}-\frac{\cos sx}{s^2}\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(0-\frac{\cos as}{s^2}\right)-\left(0-\frac{1}{s^2}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-\cos as}{s^2}+\frac{1}{s^2}\right] = \sqrt{\frac{2}{\pi}}\frac{1}{s^2}[1-\cos as]$$

$$F(s)=\sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[2\sin^2\frac{as}{2}\right] \qquad \ldots \qquad (1)$$

By inversion formula

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}\,ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\frac{1}{s^2}\sin^2\frac{as}{2}e^{-isx}\,ds$$

$$f(x)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{s^2}\sin^2\frac{as}{2}e^{-isx}\,ds$$

We have to deduce that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$ in the above integrand put $x=0$ and $a=2$ we get

$$(1)\Rightarrow \qquad f(0)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}ds$$

$$=\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 s}{s^2}ds \qquad \because \frac{\sin^2 s}{s^2}\text{ is an even function}$$

$$\int_0^{\infty}\frac{\sin^2 s}{s^2}ds=\frac{\pi}{4}f(0)=\frac{\pi}{4}[2]\left[\because f(x)=\begin{cases}a-|x|, & |x|<a\\ 0 & otherwise\end{cases}\ f(0)=a \text{ but Here a = 2}\right]$$

By Parseval's identity $\int_{-\infty}^{\infty}|f(x)|^2\,dx=\int_{-\infty}^{\infty}|F(s)|^2\,ds$

$$\int_{-a}^{a}\left|a-|x|\right|^2 dx=\int_{-\infty}^{\infty}\left|\sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[2\sin^2\frac{as}{2}\right]\right|^2 ds$$

$$2\int_0^{a}(a-x)^2 dx=\frac{2}{\pi}\times2\times4\int_0^{\infty}\frac{1}{s^4}\sin^4\frac{as}{2}ds$$

Put $a=2$ and $s=t$ we get

$$2\int_0^{2}(2-x)^2 dx=\frac{2}{\pi}\times2\times4\int_0^{\infty}\frac{1}{s^4}\sin^4\frac{2s}{2}ds$$

$$2\left(\frac{(2-x)^3}{-3}\right)_0^2=\frac{2}{\pi}\times2\times4\int_0^{\infty}\frac{1}{s^4}\sin^4 s\,ds$$

$$\frac{16}{3}=\frac{2}{\pi}\times2\times4\int_0^{\infty}\frac{1}{s^4}\sin^4 s\,ds$$

$$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$$

Answer: $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$ [$\because$ s is a dummy variable]

$$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$$

2. Find the Fourier Transform of $f(x)=\begin{cases}1-|x|, & for\ |x|<1\\ 0, & for\ |x|>1\end{cases}$ **hence deduce that**

$\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$ **,** $\int_0^\infty\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$ **[AU N/D 2015, A/M 2016, N/D 2016, N/D 2017]**

Solution: **Formula** $F[s]=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

Given $f(x)=\begin{cases}1-|x| & for\ |x|<1 \quad -1<x<1\\ 0, & otherwise\end{cases}$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}[1-|x|]\ e^{isx}\ dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}[1-|x|][\cos sx+i\sin sx]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[2\int_0^1(1-x)\cos sxdx+0\right] \because [1-|x|]\cos sx \text{ is an even function.}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1[1-x]\cos sxdx \quad \because [1-|x|]\sin sx \text{ is an odd function.}$$

$$=\sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s}-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$=\sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s}-\frac{\cos sx}{s^2}\right]_0^1$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(0-\frac{\cos s}{s^2}\right)-\left(0-\frac{1}{s^2}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-\cos s}{s^2}+\frac{1}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{s^2}[1-\cos s]$$

$$F(s)=\sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[2\sin^2\frac{s}{2}\right] \qquad \ldots \qquad (1)$$

By inversion formula

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\frac{1}{s^2}\sin^2\frac{s}{2}e^{-isx}ds$$

$$f(x)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{s^2}\sin^2\frac{s}{2}e^{-isx}ds$$

we have to deduce that $\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt$ **in the above integrand put x = 0 and a = 2 we get** $f(0)=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}ds=\frac{4}{\pi}\int_0^\infty\frac{\sin^2 s}{s^2}ds$ [$\because \frac{\sin^2 s}{s^2}$ is an even function]

$\int_0^\infty\frac{\sin^2 s}{s^2}ds=\frac{\pi}{4}f(0)$ $=\frac{\pi}{4}[2]$ [$\because f(x)=\begin{cases}1-|x|, & |x|<a\\ 0 & otherwise\end{cases}$ $f(0) = a$ but a = 2]

$$\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$$ [$\because$ s is a dummy variable]

By Parseval's identity $\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(s)|^2ds$

$$\int_{-1}^{1}\left|1-|x|\right|^2dx=\int_{-\infty}^{\infty}\left|\sqrt{\frac{2}{\pi}}\frac{1}{s^2}\left[2\sin^2\frac{s}{2}\right]\right|^2ds$$

$$2\int_0^a(1-x)^2dx=\frac{2}{\pi}\times2\times4\int_0^\infty\frac{1}{s^4}\sin^4\frac{s}{2}ds$$

Put $\frac{s}{2}=t \Rightarrow s=2t$

$ds=2dt$

$$2\int_0^1(1-x)^2dx=\frac{2}{\pi}\times2\times4\int_0^\infty\frac{1}{(2t)^4}\sin^4(t)\,2dt$$

$$2\left(\frac{(1-x)^3}{-3}\right)_0^1=\frac{2}{\pi}\times2\times4\times\frac{1}{16}\times2\int_0^\infty\frac{1}{t^4}\sin^4 t\,dt$$

$$\frac{2}{3}=\frac{2}{\pi}\int_0^\infty\frac{1}{t^4}\sin^4 t\,dt$$

$$\int_0^\infty\left(\frac{\sin t}{t}\right)^4dt=\frac{\pi}{3}$$

Result:

$$\int_0^\infty\left(\frac{\sin\ t}{t}\right)^2dt=\frac{\pi}{2},$$

$$\int_0^\infty\left(\frac{\sin t}{t}\right)^4dt=\frac{\pi}{3}$$

3. Find the Fourier transform of the function *f(x)* defined by

$$f(x)=\begin{cases} a^2-x^2 & if\,|x|<a \\ 0 & if\,|x|\geq 0 \end{cases}$$

Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt=\frac{\pi}{4}$ **. Using Parseval's Identity show that** $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt=\frac{\pi}{15}$ **.** **[A.U. April, 1996, 2000, N/D 2012,2013]**

Solution:

Given: The given function can be written as $f(x)=\begin{cases} a^2-x^2 & if\ -a<x<a \\ 0 & otherwise \end{cases}$

Formula:

$$F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left[a^2-x^2\right][\cos sx+i\sin sx]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a^2-x^2)\cos sx dx+0\right] \because \left[a^2-x^2\right]\cos sx \text{ is an even function.}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^a\left[a^2-x^2\right]\cos sxdx \quad \because (a^2-x^2)\sin sx \text{ is an odd function.}$$

$$=\sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\frac{\sin sx}{s}-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-2a}{s^2}[\cos as]+\frac{2}{s^3}(\sin as)\right]$$

$$F(S)=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$$

By Fourier inversion formula we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\frac{(\sin as - sa\cos as)}{s^3}e^{-isx}ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\ as - sa\cos\ as}{s^3}\right)\cos\ sx\ ds - i\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\ as - sa\cos\ as}{s^3}\right)\sin\ sx\ ds$$

$$f(x) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin\ as - as\cos\ as}{s^3}\cos\ sxds + 0$$

Put a=1

$$f(x) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos\ sxds$$

$$f(x) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos\ sxds$$

Put x=0 we get

$$f(0) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}ds$$

$$\frac{\pi}{4} = \int_0^{\infty}\frac{\sin s - s\cos s}{s^3}ds$$

By Parseval's identity $\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|F(s)|^2\,ds$

$$\int_{-1}^{1}(a^2 - x^2)^2\,dx = \int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right]^2 ds$$

$$2\int_0^1(a^2 - x^2)^2\,dx = \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds$$

$$\frac{16}{15} = \frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds$$

$$\frac{\pi}{15} = \int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds$$

Answer:
$$\frac{\pi}{15} = \int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt$$

4. Find the Fourier transform of the function *f(x)* defined by $f(x) = \begin{cases} 1 - x^2 & if\ |x| < 1 \\ 0 & if\ |x| \geq 1 \end{cases}$

and hence deduce that (i) $\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{16}$

(ii) $\frac{\pi}{15} = \int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$ **[AU N/D 2013, A/M 2018]**

Solution:

The given function can be written as $f(x)=\begin{cases}1-x^2 & if\ -1<x<1\\ 0 & otherwise\end{cases}$

Formula: $$F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}[1-x^2][\cos sx+i\sin sx]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[2\int_{0}^{1}(1-x^2)\cos sxdx+0\right] \quad \because [1-x^2]\cos sx \text{ is an even function.}$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{1}[1-x^2]\cos sxdx \quad \because 1-x^2\sin sx \text{ is an odd function.}$$

$$=\sqrt{\frac{2}{\pi}}\left[(1-x^2)\frac{\sin sx}{s}-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-2}{s^2}[\cos s]+\frac{2}{s^3}(\sin s)\right]$$

$$F(S)=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s-s\cos s}{s^3}\right)$$

By Fourier inversion formula we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2}{s^3}\sqrt{\frac{2}{\pi}}(\sin s-s\cos s)e^{-isx}ds$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx\,ds-i\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)\sin sx\,ds$$

Since $\frac{\sin s-s\cos s}{s^3}\cos sx$ is an even function and $\frac{\sin s-s\cos s}{s^3}\sin sx$ is an odd function we get

$$f(x)=\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin s-s\cos s}{s^3}\cos sxds+0 \;\therefore$$

$$\int_{0}^{\infty}\frac{\sin s-s\cos s}{s^3}\cos sx\;ds=\frac{\pi}{4}f(x)$$

put $x=\frac{1}{2}$, we get

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right) ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\left[1-\left(\frac{1}{2}\right)^2\right]$$

$$= \frac{\pi}{4}\left[1-\frac{1}{4}\right]$$

$$= \frac{\pi}{4}\left[\frac{3}{4}\right] = \frac{3\pi}{16}$$

By Parseval's Identity $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

$$\int_{-1}^{1}(1-x^2)^2\, dx = \int_{-\infty}^{\infty}\left[\frac{4}{\sqrt{2\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right]^2 ds$$

$$2\int_0^1 (1-x^2)^2\, dx = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{15} = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{\pi}{15} = \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

Answer: $$\frac{\pi}{15} = \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$$

5. Find the Fourier transform of $f(x)$ is given by $f(x) = \begin{cases} 1, & for\ |x| < a \\ 0, & for\ |x| > a > 0 \end{cases}$

and hence deduce that (i) $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$ **and** (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$**.**

Solution: **[AU M/J 2011, 2013, A/M 2015]**

Formula: $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx} dx$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} [\cos sx + i\sin sx]dx \qquad \because \cos sx \text{ is an even function.}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a \cos sxdx \qquad \because \sin sx \text{ is an odd function.}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^a$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)$$

(i) Using Fourier inverse formula

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx} ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)(\cos sx)\, ds - \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)(\sin sx)\, ds$$

$\left(\frac{\sin as}{s}\right)(\sin sx)$ is odd function .

There fore $\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)(\sin sx)\, ds = 0$

There fore $f(x) = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin as}{s}\right)(\cos sx)\, ds$

$$\int_0^{\infty}\left(\frac{\sin as}{s}\right)(\cos sx)\, ds = \frac{\pi}{2} f(x)$$

$$\int_0^{\infty}\left(\frac{\sin as}{s}\right)(\cos sx)\, ds = \begin{cases} \frac{\pi}{2} & when\ |x| < a \\ 0 & when\ |x| > a \end{cases}$$

In particular if $x = 0 \quad \Rightarrow \int_0^{\infty}\left(\frac{\sin as}{s}\right) ds = \frac{\pi}{2}$

$putting \quad as = t \qquad when \quad s = 0\ ,\ t = 0$

$\therefore ds = \frac{dt}{a} \qquad when\ s = \infty\ , t = \infty$

There fore
$$\int_0^{\infty}\left(\frac{\sin t}{t/a}\right)\frac{dt}{a} = \frac{\pi}{2}$$

$$\int_0^{\infty}\left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$$

(ii) **By Parseval's Identity**

$$\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|F(s)|^2 ds$$

Since $f(x) = 0$ *in* $-\infty < x < -a$ *and* $a < x < \infty$.

$$\int_{-a}^{a}(1)^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds$$

$$(x)_{-a}^{a} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds = (a+a)$$

$$\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)^2 ds = \frac{\pi a}{2} \qquad \textit{putting} \quad as = t \qquad \textit{when} \ \ s=0\ ,\ t=0$$

$$\therefore ds = \frac{dt}{a} \qquad \textit{when} \ \ s=\infty\ ,t=\infty$$

$$\therefore \int_{0}^{\infty}\left(\frac{\sin t}{t/a}\right)^2 \frac{dt}{a} = \frac{\pi\, a}{2} \ \Rightarrow \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

6. Find the Fourier transform of $f(x)$ is given by $f(x)=\begin{cases}1, & \textit{for } |x|<2\\ 0, & \textit{for } |x|>2\end{cases}$

and hence evaluate $\int_{0}^{\infty}\frac{\sin t}{t}dt$ and $\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$.

Solution: **[AU M/J 2011, 2013, A/M 2015, A/M 2017]**

Formula: $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}dx$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-2}^{2} e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2}\left[\cos sx + i\sin sx\right]dx \qquad \because \cos sx \text{ is an even function.}$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{2}\cos sx dx \qquad \because \sin sx \text{ is an odd function.}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^2$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\left(\frac{\sin 2s}{s}\right)$$

(i) Using Fourier inverse formula

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}ds$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{\sin 2s}{s}\right)(\cos sx - i\sin sx)\,ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin 2s}{s}\right)(\cos sx)\,ds-\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin 2s}{s}\right)(\sin sx)\,ds$$

$\left(\frac{\sin 2s}{s}\right)(\sin sx)$ is odd function

.

There fore $\int_{-\infty}^{\infty}\left(\frac{\sin 2s}{s}\right)(\sin sx)\,ds=0$

There fore $f(x)=\frac{2}{\pi}\int_{0}^{\infty}\left(\frac{\sin 2s}{s}\right)(\cos sx)\,ds$

$$\int_{0}^{\infty}\left(\frac{\sin 2s}{s}\right)(\cos sx)\,ds=\frac{\pi}{2}f(x)$$

$$\int_{0}^{\infty}\left(\frac{\sin 2s}{s}\right)(\cos sx)\,ds=\begin{cases}\frac{\pi}{2} & when\ |x|<2\\ 0 & when\ |x|>2\end{cases}$$

In particular if $x=0$ $\Rightarrow \int_{0}^{\infty}\left(\frac{\sin 2s}{s}\right)ds=\frac{\pi}{2}$

$putting\ \ 2s=t \qquad when\ \ s=0,\ t=0$

$\therefore ds=\frac{dt}{2} \qquad when\ \ s=\infty,\ t=\infty$

There fore $\int_{0}^{\infty}\left(\frac{\sin t}{t/2}\right)\frac{dt}{2}=\frac{\pi}{2}$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)dt=\frac{\pi}{2}$$

(ii) **By Parseval's Identity**

$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\int_{-\infty}^{\infty}|F(s)|^2\,ds$ $\quad$ *Since* $f(x)=0$ *in* $-\infty<x<-a$ *and* $a<x<\infty.$

$$\int_{-2}^{2}(1)^2\,dx=\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin 2s}{s}\right)^2 ds$$

$$(x)_{-2}^{2}=\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin 2s}{s}\right)^2 ds$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin 2s}{s}\right)^2 ds=(2+2)$$

$\int_{0}^{\infty}\left(\frac{\sin 2s}{s}\right)^2 ds=\frac{\pi 2}{2}$ $\qquad putting\ \ 2s=t \qquad when\ \ s=0,\ t=0$

$\therefore ds=\frac{dt}{2} \qquad when\ \ s=\infty,\ t=\infty$

$$\therefore \int_0^\infty \left(\frac{\sin t}{t/2}\right)^2 \frac{dt}{2} = \frac{\pi 2}{2} \Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

7. Show that the Fourier Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$

(OR) Show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform.

(OR) Find the Fourier transform of $e^{\frac{-x^2}{2}}$

[AU M/J 2013, A/M 2016, A/M 2018]

Solution:

If a transformation of a function f(x) is equal to f(s) then the function f(x) is called self reciprocal.

Given: $f(x) = e^{\frac{-x^2}{2}}$

Formula:

$$F[s] = F.[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}+isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[x^2-2isx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[(x-is)^2+(s)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-is)^2-\frac{1}{2}(s)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} e^{-\frac{s^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} dx$$

put $y = \frac{x-is}{\sqrt{2}}$ $\quad x \to -\infty \quad \Rightarrow y \to -\infty \qquad dy = \frac{1}{\sqrt{2}} dx \;\; x \to \infty \quad \Rightarrow y \to \infty$

(i.e.,) $\qquad dx = \sqrt{2} dy$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} e^{-\frac{-s^2}{2}} \sqrt{2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} 2\int_{0}^{\infty} e^{-y^2} dy \qquad \left[\because e^{-y^2} \text{ is even}\right]$$

$$= \frac{2}{\sqrt{\pi}} e^{\frac{-s^2}{2}} \left(\frac{\sqrt{\pi}}{2}\right) \qquad \left[\because \int_{0}^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}\right]$$

Answer: $= e^{-s^2/2}$

Hence $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.

8. **Find the Fourier Transform of $e^{-a^2x^2}$. Hence show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform.**
[AU M/J 2000, N/D 2014, A/M 2015, N/D 2016]

Solution:

If a transformation of a function f(x) is equal to f(s) then the function f(x) is called self reciprocal.

Given: $f(x) = e^{-a^2x^2}, a > 0$

Formula: $F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[a^2x^2 - isx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \left(\frac{s^2}{4a^2}\right)\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[ax - \frac{is}{2a}\right]^2} e^{-\frac{1}{4a^2}s^2} dx$$

Put $u = ax - \frac{is}{2a}$ $x \to -\infty \quad \Rightarrow u \to -\infty \qquad x \to \infty \quad \Rightarrow u \to \infty$

(i.e.,) $du = adx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} e^{\frac{-s^2}{4a^2}} \frac{1}{a} du$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2} e^{\frac{-s^2}{4a^2}} \frac{1}{a} du$$

$$\left[\because \int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2} \right]$$

$$= \frac{2e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \int_0^\infty e^{-u^2} du$$

$$= \frac{2e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} \frac{\sqrt{\pi}}{2} = \frac{e^{-\frac{s^2}{4a^2}}}{a} \frac{1}{\sqrt{2}}, \ put \quad a = \frac{1}{\sqrt{2}}$$

Answer: $$= e^{-\frac{s^2}{2}}$$

COSINE TRANSFORM

FORMULA **(1)** $F_c[s] = F[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$

(2) $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, ds$

9. Show that the Fourier cosine Transform of $e^{-a^2x^2}$ **(OR) Show that** $e^{\frac{-x^2}{2}}$ **is self – reciprocal with respect to Fourier Transform.**

[AU N/D 1996,2012 M/J 2000, A/M 2017]

Solution:

If a transformation of a function f(x) is equal to f(s) then the function f(x) is called self reciprocal.

Given: $f(x) = e^{-a^2x^2}, a > 0$

Formula: $F_c[s] = F.[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$

$$= \frac{1}{\sqrt{2\pi}} RP\, of \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} RP\, of \int_{-\infty}^{\infty} e^{-\left[a^2x^2 - isx\right]} \quad dx$$

$$= \frac{1}{\sqrt{2\pi}} RP \, of \int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} RP \, of \int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2 + \left(\frac{s^2}{4a^2}\right)\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} RP \, of \int_{-\infty}^{\infty} e^{-\left[ax-\frac{is}{2a}\right]^2} e^{-\frac{1}{4a^2}s^2} dx$$

Put $u = ax - \frac{is}{2a}$ $\quad x \to -\infty \quad \Rightarrow u \to -\infty \quad x \to \infty \quad \Rightarrow u \to \infty$

(i.e.,) $\quad du = adx$

$$= \frac{1}{\sqrt{2\pi}} RP \, of \int_{-\infty}^{\infty} e^{-u^2} e^{\frac{-s^2}{4a^2}} \frac{1}{a} du$$

$$= \frac{2}{\sqrt{2\pi}} RP \, of \int_{0}^{\infty} e^{-u^2} e^{\frac{-s^2}{4a^2}} \frac{1}{a} du$$

$$= \frac{2e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} RP \, of \int_{0}^{\infty} e^{-u^2} du$$

$\left[\because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}\right]$

$$= \frac{2e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} RP \, of \, \frac{\sqrt{\pi}}{2} = \frac{e^{-\frac{s^2}{4a^2}}}{a} \frac{1}{\sqrt{2}}, \; put \quad a = \frac{1}{\sqrt{2}}$$

Answer: $\quad = e^{-\frac{s^2}{2}}$

10. Show that the Fourier cosine Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$ (OR) Show that $e^{\frac{-x^2}{2}}$ is self –reciprocal with respect to Fourier Transform.

[A.U. N/D 1996, M/J - 2000]

Solution:

If a transformation of a function f(x) is equal to f(s) then the function f(x) is called self reciprocal.

Given: $f(x) = e^{\frac{-x^2}{2}}$

Formula $F[s] = F.[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$

$$F[s] = F[f(x)] = \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} \cos sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}+isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[x^2-2isx\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[x^2-2isx+(is)^2-(is)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left[x^2-2isx+(is)^2\right]} e^{\frac{1}{2}(is)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} e^{-\frac{s^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} e^{-\frac{-s^2}{2}} dx$$

put $y = \frac{x-is}{\sqrt{2}}$ $x \to -\infty$ $\Rightarrow y \to -\infty$ $dy = \frac{1}{\sqrt{2}} dx$ $x \to \infty$ $\Rightarrow y \to \infty$

(i.e.,) $dx = \sqrt{2} dy$ $= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-y^2} e^{-\frac{-s^2}{2}} \sqrt{2} dy$

$$= \frac{1}{\sqrt{\pi}} R.P. e^{-\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{--s^2/2} R.P. 2\int_{0}^{\infty} e^{-y^2} dy \quad \left[\because e^{-y^2} \text{ is even}\right]$$

$$= \frac{2}{\sqrt{\pi}} R.P. e^{--s^2/2}\left(\frac{\sqrt{\pi}}{2}\right) \quad \left[\because \int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}\right]$$

Answer: $= e^{-s^2/2}$

11. Find the Fourier cosine transform of e^{-x^2} [A.U. N / D 2004]

Solution:

Formula $F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$

$$F_c\left[e^{-x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2} \cos sx\, dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\int_{-\infty}^{\infty}e^{-x^2}\cos sx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}R..P.\int_{-\infty}^{\infty}e^{-x^2}e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}R..P.\int_{-\infty}^{\infty}e^{-x^2+isx}d\,x$$

$$=\frac{1}{\sqrt{2\pi}}R..P.\int_{-\infty}^{\infty}e^{-\left(x^2-isx\right)}d\,x$$

$$=\frac{1}{\sqrt{2\pi}}R..P.\int_{-\infty}^{\infty}e^{-\left[\left(x-\frac{is}{2}\right)^2+\frac{s^2}{4}\right]}d\,x$$

$$=\frac{1}{\sqrt{2\pi}}R..P.\int_{-\infty}^{\infty}e^{-\left[x--\frac{is}{2}\right]^2}e^{\frac{-s^2}{4}}d\,x$$

$$=\frac{1}{\sqrt{2\pi}}e^{\frac{-s^2}{4}}R..P.\int_{-\infty}^{\infty}e^{-\left[x--\frac{is}{2}\right]^2}d\,x$$

put $t=x-\frac{is}{2}$ $\qquad x\to-\infty\Rightarrow t\to-\infty$

dt = dx $\qquad x\to\infty \qquad \Rightarrow t\to\infty$

$$=\frac{1}{\sqrt{2\pi}}e^{\frac{-s^2}{4}}R..P.\int_{-\infty}^{\infty}e^{-[t]^2}dt$$

$$=\frac{1}{\sqrt{2\pi}}e^{\frac{-s^2}{4}}R..P.\int_{-\infty}^{\infty}e^{-t^2}dt$$

$$=\frac{1}{\sqrt{2\pi}}e^{\frac{-s^2}{4}}R..P.\sqrt{\pi} \qquad \because\int_{-\infty}^{\infty}e^{-t^2}dt=\sqrt{\pi}$$

$$=\frac{1}{\sqrt{2}}e^{\frac{-s^2}{4}}$$

Answer:

$$F_c\left[e^{-x^2}\right]=\frac{1}{\sqrt{2}}e^{\frac{-s^2}{4}}$$

12. Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$ and hence find $F_C\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$

Solution: **[AU NOV 2015]**

Formula $F_C\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_C\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$$

$$F_C[s]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$$

Diff .w.r.t 's'

$$\frac{dF_C[s]}{ds}=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx\right]$$

$$\frac{dF_C[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}\cos sx\,dx\right]$$

$$\frac{dF_C[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}(-\sin sx)x\,dx\right]$$

$$\frac{dF_C[s]}{ds}=-\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sxdx\right]$$

$$\frac{dF_C[s]}{ds}=-\left[\sqrt{\frac{2}{\pi}}\ \frac{s}{a^2+s^2}\right]$$

$$\int\frac{dF_C[s]}{ds}=-\left[\sqrt{\frac{2}{\pi}}\int\frac{s}{a^2+s^2}ds\right]$$

$$\int\frac{dF_C[s]}{ds}=-\left[\sqrt{\frac{2}{\pi}}\frac{1}{2}\int\frac{2s}{a^2+s^2}ds\right]$$

$$\int\frac{dF_C[s]}{ds}=-\left[\sqrt{\frac{2}{\pi}}\frac{1}{2}\log\left(s^2+a^2\right)\right]$$

$$F_C[s]=-\left[\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)\right]$$

Similarly, $F_C\left[\frac{e^{-bx}}{x}\right]=-\left[\frac{1}{\sqrt{2\pi}}\log\left(s^2+b^2\right)\right]$

$$F_C\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]=F_C\left[\frac{e^{-ax}}{x}\right]-\left[\frac{e^{-bx}}{x}\right]$$

$$=-\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)+\frac{1}{\sqrt{2\pi}}\log\left(s^2+b^2\right)$$

Answer: $F_C\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]=\frac{1}{\sqrt{2\pi}}\log\frac{\left(s^2+b^2\right)}{\left(s^2+a^2\right)}$

13. Find the FCT of $e^{-ax}\cos ax$

Solution:

Formula:

$$F_C\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_C\left[e^{-ax}\cos ax\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos ax\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty e^{-ax}[\cos(s+a)x+\cos(s-a)x]dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_0^\infty e^{-ax}\cos(s+a)xdx+\int_0^\infty e^{-ax}\cos(s-a)xdx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{a}{a^2+(s+a)^2}+\frac{a}{a^2+(s-a)^2}\right]$$

$$=\frac{a}{\sqrt{2\pi}}\left[\frac{a^2+(s-a)^2+a^2+(s+a)^2}{[a^2+(s+a)^2]\{a^2+(s-a)^2\}}\right]$$

$$=\frac{a}{\sqrt{2\pi}}\left[\frac{4a^2+2s^2}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right]$$

Answer: $F_C\left[e^{-ax}\cos ax\right]=\frac{2a}{\sqrt{2\pi}}\left[\frac{2a^2+s^2}{s^4+4a^4}\right]$

14. Find the FCT of $e^{-ax}\sin ax$

Solution:

Formula: $F_C\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_C\left[e^{-ax}\cos ax\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin ax\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty e^{-ax}[\sin(s+a)x-\sin(s-a)xdx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_0^\infty e^{-ax}[\sin(s+a)xdx-\int_0^\infty \sin(s-a)x\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{s+a}{a^2+(s+a)^2}-\frac{s-a}{a^2+(s-a\ )^2}\right]$$

$$=\frac{a}{\sqrt{2\pi}}\left[\frac{2a^2+s^2-2as\ )(s+a)}{2a^2+s^2-2as\ )}-\frac{(2a^2+s^2+2as\ )(s-a)}{(2a^2+s^2-2as\ )}\right]$$

$$=\frac{a}{\sqrt{2\pi}}\left[\frac{4a^3-2as^2}{(2a^2+s^2)^2-(2as)^2}\right]$$

Answer:

$$F_C\left[e^{-ax}\sin ax\right]=\frac{2a}{\sqrt{2\pi}}\left[\frac{2a^2-s^2}{s^4+4a^4}\right]$$

SINE TRANSFORM

Formula

$$F_S\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,ds$$

15. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$ and hence find $F_S\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$

Solution: **[AU N/D 2011, N/D 2016]**

Formula: $F_S\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_S\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

$$F_S\left[s\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

Diff .w.r.t ‘s’

$$\frac{dF_S[s]}{ds}=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx\right]$$

$$\frac{dF_S[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{d}{ds}\sin sx\,dx\right]$$

$$\frac{dF_S[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(\cos sx)x dx\right]$$

$$\frac{dF_s[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx dx\right]$$

$$\frac{dF_S[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\;\frac{a}{a^2+s^2}\right]$$

$$\int\frac{dF_S[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\int\frac{a}{a^2+s^2}ds\right]$$

$$\int\frac{dF_S[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}\int\frac{a}{a^2+s^2}ds\right]$$

$$\int\frac{dF_S[s]}{ds}=\left[\sqrt{\frac{2}{\pi}}a\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)+c\right]$$

$$F_S\left[\frac{e^{-ax}}{x}\right]=\left[\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)+c\right]$$

Similarly $F_S\left[\frac{e^{-bx}}{x}\right]=\left[\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)+c\right]$

Answer:

$$F_S\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]=F_S\left[\frac{e^{-ax}}{x}\right]-F_S\left[\frac{e^{-bx}}{x}\right]=\left[\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)\right]-\left[\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right)-\tan^{-1}\left(\frac{s}{b}\right)\right]$$

16. Find f(x), if its sine transform is $\frac{e^{-sa}}{s}$. Hence find reciprocal of $\frac{1}{s}$.

Solution: **[A.U. N/D 2013]**

Formula: $$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_S(s)\sin sx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds$$

Diff .w.r.t 'x'

$$\frac{d[f(x)]}{dx}=\frac{d}{dx}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds\right]$$

$$\frac{d[f(x)]}{dx}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-sx}}{s}\frac{d}{dx}\sin sx\, ds\right]$$

$$\frac{d[f(x)]}{dx}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}(\cos sx)s\, ds\right]$$

$$\frac{d[f(x)]}{dx}=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\cos sxds\right]$$

$$\frac{d[f(x)]}{dx}=\left[\sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+x^2}\right]$$

$$\int\frac{d[f(x)]}{dx}=\left[\sqrt{\frac{2}{\pi}}\int\frac{a}{a^2+x^2}dx\right]$$

$$\int\frac{d[f(x)]}{dx}=\left[\sqrt{\frac{2}{\pi}}\int\frac{a}{a^2+x^2}dx\right]$$

$$f(x)=\left[\sqrt{\frac{2}{\pi}}a\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$F_S^{-1}\left[\frac{e^{-ax}}{s}\right]=\left[\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)\right]\text{'=======(1)}$$

To find the F.S.T of $\frac{1}{s}$

Put a=0 in (1) $F_S^{-1}\left[\frac{e^{-ax}}{s}\right]=\left[\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{0}\right)\right]$

$$F_S^{-1}\left[\frac{e^{-ax}}{s}\right]=\left[\sqrt{\frac{2}{\pi}}\tan^{-1}(\infty)\right]=\sqrt{\frac{2}{\pi}}.\frac{\pi}{2}=\sqrt{\frac{\pi}{2}}$$

Answer:

SINE AND COSINE TRANSFORMS

17. (1) Find the FCT of $\frac{1}{x^2+1}$ **(2) Find the FST of** $\left[\frac{x}{x^2+1}\right]$

Solution

Formula: $F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_C\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx$$

$$F_C[s] = \sqrt{\frac{2}{\pi}}\frac{1}{s^2+1}$$

$$f(x) = \left[\sqrt{\frac{2}{\pi}}\int_0^\infty F_C[f(x)]\cos sx\,ds\right]$$

$$f(x) = \left[\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{s^2+1}\cos sx\,ds\right]$$

$$e^{-x} = \left[\frac{2}{\pi}\int_0^\infty \frac{1}{s^2+1}\cos sx\,ds\right]$$

$$\frac{\pi}{2}e^{-x} = \left[\int_0^\infty \frac{1}{s^2+1}\cos sx\,ds\right]$$

$$\frac{\pi}{2}e^{-s} = \left[\int_0^\infty \frac{1}{s^2+1}\cos sx\,ds\right] \qquad x = s$$

$$\sqrt{\frac{2}{\pi}}\frac{\pi}{2}e^{-s} = \sqrt{\frac{2}{\pi}}\left[\int_0^\infty \frac{1}{s^2+1}\cos sx\,ds\right]$$

Answer:

$$F_C\left[\frac{1}{x^2+1}\right] = \left[\sqrt{\frac{\pi}{2}}e^{-s}\right]$$

(2) Fourier sine transform of $\frac{x}{x^2+1}$

Formula: $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_S\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx$$

$$F_C[s] = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+1}$$

$$f(x) = \left[\sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds\right]$$

$$f(x)=\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{s^2+1}\sin sx\,ds\right]$$

$$e^{-x}=\left[\frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1}\sin sx\,ds\right]$$

$$\frac{\pi}{2}e^{-s}=\left[\int_0^\infty \frac{x}{x^2+1}\sin sx\,dx \qquad s=x\right]$$

$$\sqrt{\frac{2}{\pi}}\frac{\pi}{2}e^{-s}=\sqrt{\frac{2}{\pi}}\left[\int_0^\infty \frac{x}{x^2+1}\sin sx\,dx\right]$$

Answer:

$$F_S\left[\frac{x}{x^2+1}\right]=\left[\sqrt{\frac{\pi}{2}}e^{-s}\right]$$

18. Find the Fourier cosine transform of $f(x)=e^{-a^2x^2}$ and hence find the Fourier cosine transform of $e^{\frac{-x^2}{2}}$ and Fourier sine transform of $xe^{\frac{-x^2}{2}}$.

Solution:

Formula: **[A.U. N/D 2006]**

$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cos sx dx \Rightarrow \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx dx \Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2} R.P of e^{isx} dx$$

$$=R.P.of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx}dx$$

$$=R.P.of\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx}\ \frac{e^{s^2/4a^2}}{e^{s^2/4a^2}}dx$$

$$=R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx+s^2/4a^2}dx$$

$$=R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(ax-is/2a\right)^2}dx$$

$$put\ \ ax-{is}/{2a}=y \qquad when\ x=-\infty\ ,\ y=-\infty$$

$$adx=dy \qquad x=\infty\ ,\ y=\infty$$

$$= R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(y)^2}\ \frac{dy}{a} \Rightarrow R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\sqrt{\pi}$$

$$F_c[f(x)] = \frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$a^2 = \frac{1}{2}$$

Put $F_c[e^{\frac{-x^2}{2}}] = \dfrac{e^{-s^2/4(\frac{1}{4})}}{\sqrt{2} \times \frac{1}{\sqrt{2}}} = e^{\frac{-s^2}{2}}$

$$F_c[e^{\frac{-x^2}{2}}] = e^{\frac{-s^2}{2}}$$

$$(ii)\quad F_s[xf(x)] = -\frac{d(F_c[f(x)])}{ds}$$

$$F_s[xe^{\frac{-s^2}{2}}] = -\frac{d}{ds}\left[e^{\frac{-s^2}{2}}\right]$$

$$= -\left[e^{-s^2/2}\left(\frac{-2s}{2}\right)\right]$$

Answer :

$$\therefore\ F_s[xe^{\frac{-s^2}{2}}] = s\,e^{-s^2/2} \qquad [\because S\ is\ dummy\ \ Variable]$$

19. Prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform and cosine .

Solution **[AU April, 1996, 2000, M/J 2012, Apr/May 2015]**

To prove: $F_C\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$

Formula: $F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$

$$F_C\left[x^{n-1}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} x^{n-1}\cos sx\,dx$$

w. k. t $\Gamma n = \int_0^{\infty} e^{-x}x^{n-1}\,dx$

$$x = at$$

$$dx = adt \quad x = 0, x = \infty, t = 0, t = \infty$$

$$\Gamma n = \int_0^{\infty} e^{-at}(at)^{n-1} a\,dt$$

$$\Gamma n = \int_0^\infty e^{-at}(t)^{n-1} a^n \, dt$$

$$\frac{\Gamma n}{a^n} = \int_0^\infty e^{-at}(t)^{n-1} \, dt$$

$$\frac{\Gamma n}{(is)^n} = \int_0^\infty e^{-isx}(x)^{n-1} \, dx \qquad a = is$$

$$\frac{\Gamma n}{(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^n s^n} = \int_0^\infty (\cos sx - i\sin sx)(x)^{n-1} \, dx$$

$$\frac{\Gamma n \quad (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^{-n}}{s^n} = \int_0^\infty (\cos sx - i\sin sx)(x)^{n-1} \, dx$$

Equating Real part

$$\frac{\Gamma n\,(\cos n\frac{\pi}{2} - i\sin n\frac{\pi}{2})}{s^n} = \int_0^\infty (\cos sx - i\sin sx)(x)^{n-1} \, dx$$

$$\frac{\Gamma n\,(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})^n}{s^n} = \int_0^\infty (\cos sx - i\sin sx)(x)^{n-1} \, dx$$

Equating Real part

$$\sqrt{\frac{2}{\pi}}\,\frac{\Gamma n\,(\cos n\frac{\pi}{2})}{s^n} = \sqrt{\frac{2}{\pi}}\int_0^\infty (\cos sx)(x)^{n-1} \, dx$$

$$\sqrt{\frac{2}{\pi}}\,\frac{\Gamma n(\cos n\frac{\pi}{2})}{s^n} = F_C[x^{n-1}] \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Put n=1/2 in (1)

$$\sqrt{\frac{2}{\pi}}\,\frac{\Gamma\frac{1}{2}(\cos\frac{1}{2}\frac{\pi}{2})}{s^{\frac{1}{2}}} = F_C[x^{\frac{1}{2}}]$$

$$\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}\frac{1}{\sqrt{2}} = F_C[x^{\frac{1}{2}}]$$

$$\frac{1}{\sqrt{s}} = F_C[x^{\frac{1}{2}}]$$

To prove that sine transform of $\frac{1}{\sqrt{x}}$

$$\frac{\Gamma n\,(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})^n}{s^n} = \int_0^\infty (\cos sx - i\sin sx)(x)^{n-1} \, dx$$

Equating imaginary part

$$\sqrt{\frac{2}{\pi}}\frac{\Gamma n\,(\sin n\frac{\pi}{2})}{s^n}=\sqrt{\frac{2}{\pi}}\int_0^\infty (\sin sx)(x)^{n-1}\,dx$$

$$\sqrt{\frac{2}{\pi}}\frac{\Gamma n\,(\sin n\frac{\pi}{2})}{s^n}=F_S[x^{n-1}]=======(1)$$

Put n=1/2 in (1)

$$\sqrt{\frac{2}{\pi}}\frac{\Gamma\frac{1}{2}\,(\sin\frac{1}{2}\frac{\pi}{2})}{s^{\frac{1}{2}}}=F_S[x^{\frac{1}{2}}]$$

$$\sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}\frac{1}{\sqrt{2}}=F_S[x^{\frac{1}{2}}]$$

Answer:

$$\frac{1}{\sqrt{s}}=F_S[x^{\frac{1}{2}}]$$

20. Fourier sine transform of e^{-ax} and hence find FCT of xe^{-ax}

[AU A/M 2000, M/J 2012]

Solution:

Formula: $F_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_S\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

$To\ find\ \ F_C[xe^{-ax}]$

$$by\ property\ F_C[xe^{-ax}]=\frac{d}{ds}[F_S[f(x)]$$

$$F_C[xe^{-ax}]=\frac{d}{ds}[F_S[e^{-ax}]$$

$$F_C[xe^{-ax}]=\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{\left(s^2+a^2\right)-s.2s}{\left(s^2+a^2\right)^2}$$

Answer:

$$=\sqrt{\frac{2}{\pi}}\frac{\left(a^2-s^2\right)}{\left(s^2+a^2\right)^2}$$

21. Find the Fourier Cosine transform of $f(x)=e^{-a^2x^2}$ and hence Fourier sine transform of $xe^{-a^2x^2}$ **[AU A/M 2007, N/D 2010, A/M 2018]**

Solution

Formula:
$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\,\cos sx\,dx \Rightarrow \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_{-\infty}^\infty e^{-a^2x^2}\,\cos sx\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\,\cos sx\,dx \Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\,R.P\,of\,e^{isx}\,dx$$

$$=R.P.of\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx}\,dx$$

$$=R.P.of\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx}\,\frac{e^{s^2/4a^2}}{e^{s^2/4a^2}}$$

$$=R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx+s^2/4a^2}\,dx$$

$$=R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(ax-is/2a\right)^2}\,dx$$

$$put\ \ ax-is/2a=y \qquad when\ x=-\infty,\ y=-\infty$$
$$adx=dy \qquad x=\infty,\ y=\infty$$

$$=\ R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(y)^2}\,\frac{dy}{a} \Rightarrow\ R.P\ of\ \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}}\sqrt{\pi}$$

$$F_c[f(x)]=\frac{e^{-s^2/4a^2}}{a\sqrt{2}}$$

$$F_s[xf(x)]=-\frac{d\left(F_c[f(x)]\right)}{ds}$$

Answer:
$$F_s[xe^{-a^2x^2}]=-\frac{d}{ds}\left[\frac{e^{-s^2/4a^2}}{a\sqrt{2}}\right]$$

$$F_s[xe^{-a^2x^2}]=-\frac{1}{a\sqrt{2}}\left[e^{-s^2/4a^2}\left(\frac{-2s}{4a^2}\right)\right]=\frac{s}{a^3\,2\sqrt{2}}\,e^{-s^2/4a^2}$$

Parseval's identity in F.S.T and F.C.T

FORMULA

(1) $\int_0^\infty f(x)g(x)dx=\int_0^\infty F_c[f(x)]F_c[g(x)]ds$

(2) $\int_0^\infty |f(x)|^2 dx = \int_0^\infty F_c[f(x)]^2 ds$

(3) $\int_0^\infty |f(x)|^2 dx = \int_0^\infty F_S[f(x)]^2 ds$

22. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ **using Fourier transforms**

[A.U M/J 2006, N/D 2014]

Solution

By Parseval's Identity $\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c[f(x)]F_c[g(x)]ds$ …………..(1)

Let $f(x) = e^{-ax}$	$g(x) = e^{-bx}$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$ $= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$	$F_c[g(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx\,dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\cos sx\,dx$ $= \sqrt{\frac{2}{\pi}}\left[\frac{b}{s^2+b^2}\right]$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty e^{-(a+b)x}dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \left(0 + \frac{1}{(a+b)}\right)$$

$$F_c[f(x)]F_c[g(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{b}{s^2+b^2}\right]$$

$$= \frac{2}{\pi}\ \frac{ab}{(s^2+a^2)(s^2+b^2)}$$

$$\textit{from}\ \ (1) \Rightarrow \frac{1}{a+b} = \frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds$$

$$\Rightarrow \int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds = \frac{\pi}{2ab(a+b)}$$

Answer :

$$\Rightarrow \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2ab(a+b)} \qquad [\because s \textit{ is dummy variable}]$$

23. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ **using transforms. [AU N/D 2010, A/M 2017]**

Solution

By Parseval's Identity, $\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c[f(x)]F_c[g(x)]ds$ ………….(1)

Let $f(x)=e^{-x}$	$g(x)=e^{-2x}$
$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1^2}\right]$	$F_c[g(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+2^2}\right]$

$$f(x)g(x)=e^{-ax}e^{-bx}=e^{-(a+b)x}$$

$$\int_0^\infty f(x)g(x)dx=\int_0^\infty e^{-(1+2)x}dx$$

$$=\left[\frac{e^{-(1+2)x}}{-(1+2)}\right]_0^\infty=\left(0+\frac{1}{(1+2)}\right)$$

$$F_c[f(x)]F_c[g(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+2^2}\right]$$

$$=\frac{2}{\pi}\,\frac{1*2}{(s^2+1^2)(s^2+2^2)}$$

$$from\ (1) \Rightarrow \frac{1}{1+2}=\frac{4}{\pi}\int_0^\infty \frac{1}{(s^2+1^2)(s^2+2^2)}ds$$

$$\Rightarrow \int_0^\infty \frac{1}{(s^2+1^2)(s^2+2^2)}ds=\frac{\pi}{4(1+2)}$$

$$\Rightarrow \int_0^\infty \frac{1}{(x^2+1^2)(x^2+2^2)}dx=\frac{\pi}{12} \qquad [\because s\ is\ dummy\ variable]$$

Answer: $\int_0^\infty \frac{1}{(x^2+1^2)(x^2+2^2)}dx=\frac{\pi}{12}$

24. Evaluate using transforms $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ **. [A.U. M/J 2013, N/D 2013]**

Solution

By Parseval's Identity

$$\int_0^\infty |f(x)|^2dx=\int_0^\infty |F_c[f(x)]|^2ds$$

Let $f(x) = e^{-ax}$
$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$ $= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty e^{-2ax}\,dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \frac{1}{2a}$$

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty F_c[f(x)]^2\,ds$$

$$\left(\frac{1}{2a}\right) = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]\right)^2 ds$$

$$\int_0^\infty \frac{ds}{\left(s^2+a^2\right)^2} = \frac{\pi}{4a^3}$$

Answer: $$\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{4a^3}$$

25. Evaluate using transforms $\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2}$ **.** **[A.U. N/D.2009]**

Solution

By Parseval's Identity

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty F_s[f(x)]^2\,ds \quad \ldots\ldots\ldots\ldots\text{. (1)}$$

Let $f(x) = e^{-ax}$
$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$ $= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty e^{-2ax} dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \frac{1}{2a}$$

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty F_s[f(x)]^2 ds$$

$$\left(\frac{1}{2a}\right) = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]\right)^2 ds$$

$$\int_0^\infty \frac{s^2 ds}{(s^2+a^2)^2} = \frac{\pi}{4a}$$

Answer: $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$

26. State and Prove Convolution theorem.

Statement:

The fourier transform of the convolution of f(x) and g(x) is equal to the poduct of their Fourier transforms.

(i.e.,) $F[f(x)*g(x)] = F[f(x)]F[g(x)]$

Proof:

Formula:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cos sx\, dx \Rightarrow \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx\, dx$$

$$F[s] = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{isx}dx$$

$$F[s] = F[f(x)*g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty [f(x)*g(x)]e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty [f(t)g(x-t)dt]\, e^{isx}dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^\infty\int_{-\infty}^\infty [f(t)g(x-t)]\, e^{isx}dtdx$$

By changing order of integration, we get

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty}[g(x-t)e^{isx}dx]dt$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} f(t)\left[\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}[g(x-t)e^{isx}dx]dt\right]$$

$$= \left[\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}[F[g(x-t)]]\, f(t)\; dt\right]$$

$$= \left[\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} f(t)e^{ist}G(s)\; dt\right] \quad \textbf{By property}$$

$$= G(s)\left[\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} f(t)e^{ist}\; dt\right]$$

$$= G(s).F(s)$$

$$F[f(x)*g(x)] = F[f(x)]F[g(x)]$$

27. State and Prove Parseval's Identity. [A.U. 2010, M/J 2012]

STATEMENT:

If F(S) is the Fourier transform of f(x), then $\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|F(s)|^2\; ds$

Proof:

By convolution Theorem

$$F[f(x)*g(x)] = F[S].G[S]$$

$$[f(x)*g(x)] = F^{-1}[F(s).G(s)]$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} g(x-t)f(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s).G(s)e^{-isx}ds$$

Put x=0, we get

$$\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} f(t)g(-t)dt = \left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} F(s)G(s)ds \;\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$g(-t) = \overline{f(t)}$$

$$G(s) = \overline{F(s)}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} f(t)\overline{f(t)}dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)}ds$$

$$\int_{-\infty}^{\infty}|f(t)|^2 dt = \int_{-\infty}^{\infty}|F(s)|^2\; ds$$

Answer: $$\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|F(s)|^2\; ds$$

28. Find the Fourier Cosine and Sine transform of $f(x) = e^{-ax}, a > 0$ **and hence deduce the inversion formula** (OR) **Find the Fourier Cosine and Sine**

transform of $f(x)=e^{-ax}, a>0, x\ge 0$. H**ence deduce integrals** $\int_0^\infty \frac{\cos sx}{s^2+a^2}ds$ **and** $\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$ **[A.U N/D 2012. MAY - 2016]**

Solution:

Formula: $F_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos xdx$

$$F_c\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos xdx \qquad \left[\because \int_0^\infty e^{-ax}\cos bxdx=\frac{a}{a^2+b^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$$

Using inversion formula,

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_C\left[e^{-ax}\right]\cos sxds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\cos sxds$$

$$=\frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{s^2+a^2}ds$$

$$\int_0^\infty \frac{\cos sx}{s^2+a^2}ds=\frac{\pi}{2a}f(x)$$

$$=\frac{\pi}{2a}e^{-ax}, a>0$$

$$F_S\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin xdx$$

$$F_S\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sxdx \qquad \left[\because \int_0^\infty e^{-ax}\sin bxdx=\frac{b}{a^2+b^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

Using inversion formula,

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_S\left[e^{-ax}\right]\sin sxds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sin sxds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

$$\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds = \frac{\pi}{2}f(x)$$

Answer:

$$= \frac{\pi}{2}e^{-ax}, a>0$$

29. Verify convolution theorem for $f(x)=g(x)=e^{-x^2}$ [A.U. M/J 2013,NOV 2015]

Solution.

$$F[f(x)*g(x)] = F\{f(x)\}F\{g(x)\}$$

Given: $f(x)=g(x)=e^{-x^2}$

Formula:, $F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-isx}dx$

$$F\left[e^{-x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2}\, e^{-isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(x^2+isx\right)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\left(x-\frac{is}{2}\right)^2+\frac{s^2}{4}\right]}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(x-\frac{is}{2}\right)^2} e^{-\frac{s^2}{4}}dx$$

$$F\left[e^{-x^2}\right] = e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-\left(x-\frac{is}{2}\right)^2}dx$$

$put\ t = x-\frac{is}{2}$ $\quad x\to\infty \Rightarrow t\to-\infty$

$dt = dx$ $\quad x\to\infty \Rightarrow t\to\infty$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right)2\int_0^{\infty} e^{-t^2}dt$$

$put\ t^2 = u$ $\quad t\to 0 \Rightarrow u\to 0$

$2t\ dt = du$ $\quad t\to\infty \Rightarrow u\to\infty$

$dt = \frac{1}{2t}du$

$$dt = \frac{1}{2\sqrt{u}}\,du$$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right) 2\int_0^{\infty} e^{-u}\,\frac{1}{2\sqrt{u}}\,du$$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right)\int_0^{\infty} e^{-u}\,u^{-\frac{1}{2}}\,du$$

$$= e^{-\frac{s^2}{4}}\left(\frac{1}{\sqrt{2\pi}}\right)\sqrt{\pi} \qquad \left[\because \int_0^{\infty} e^{-t}t^{-\frac{1}{2}}dt = \sqrt{\pi}\right]$$

$$F\{f(x)\} = \frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}$$

$$F\{f(x)\} = F\{g(x)\} = \frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}$$

$$\therefore F\{f(x)\}.F\{g(x)\} = \left(\frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}\right)\left(\frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}\right) = \frac{1}{2}e^{-\frac{s^2}{2}} \text{-------------}(1)$$

By convolution definition,

$$f(x)*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)g(x-u)du$$

$$e^{-x^2}*e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}e^{-(x-u)^2}du$$

$$e^{-x^2}*e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-u)^2-u^2}du$$

$$e^{-x^2}*e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left[\left(u-\frac{x}{2}\right)^2+\frac{x^2}{4}\right]}du$$

$$e^{-x^2}*e^{-x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2}e^{-\frac{x^2}{2}}du$$

$$e^{-x^2}*e^{-x^2} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\left(u-\frac{x}{2}\right)^2}du$$

$$put\ t = u-\frac{x}{2} \qquad u\to-\infty \Rightarrow t\to-\infty$$

$$dt = du \qquad u\to\infty \Rightarrow t\to\infty$$

$$e^{-x^2}*e^{-x^2} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2t^2}dt$$

$$e^{-x^2}*e^{-x^2} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}2\int_0^{\infty} e^{-2t^2}dt$$

$$put\ t^2 = y \qquad t \to 0 \Rightarrow y \to 0$$
$$2tdt = dy \qquad t \to \infty \Rightarrow y \to \infty$$
$$dt = \frac{dy}{2\sqrt{y}}$$

$$e^{-x^2} * e^{-x^2} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2t^2} dt$$

$$e^{-x^2} * e^{-x^2} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} 2\int_{0}^{\infty} e^{-2y} \frac{dy}{2\sqrt{y}}$$

$$e^{-x^2} * e^{-x^2} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2y} y^{-\frac{1}{2}} dy$$

$$e^{-x^2} * e^{-x^2} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \qquad \left[\because \int_{0}^{\infty} e^{-2y} y^{-\frac{1}{2}} dy = \sqrt{\frac{\pi}{2}}\right]$$

$$e^{-x^2} * e^{-x^2} = \frac{1}{2} e^{-\frac{x^2}{2}}$$

$$F\left[f(x) * g(x)\right] = F\left[\frac{1}{2} e^{-\frac{x^2}{2}}\right] = \frac{1}{2} F\left[e^{-\frac{x^2}{2}}\right]$$

We know that $e^{-\frac{x^2}{2}}$ is self reciprocal under Fourier transform.

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}}$$

Answer:

$$\because F\left[f(x) * g(x)\right] = \frac{1}{2} e^{-\frac{s^2}{2}} \text{ --------------- } (2)$$

we conclude that (1)=(2)

Hence convolution theorem is verified.

30. Express the function $f(x) = \begin{cases} 1 & for|x| \le 1 \\ 0 & for|x| > 1 \end{cases}$ **as a Fourier Integral. Hence evaluate** $\int_{0}^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$ **and find the value of** $\int_{0}^{\infty} \frac{\sin \lambda}{\lambda} d\lambda$ **[A U, April, 2001]**

Solution.

Formula for *f(x)* is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) e^{is(t-x)} dt\ ds$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\ ds \qquad ..(1)$$

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \quad dx \quad \text{----} \ (2)$$

Here $f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0 & otherwise \end{cases}$

$$\therefore \qquad (2) \Rightarrow \ F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1) \ e^{isx} \ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} [\cos sx + i \sin sx] dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{1} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} \right]$$

$$(1) \Rightarrow \ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2/\pi} \ \frac{\sin s}{s} e^{-isx} \ ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} [\cos sx - i \sin sx] ds$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin s}{s} \cos sx ds$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \lambda}{\lambda} \cos \lambda x d\lambda \quad \because s \, is \, dummy \quad \text{var} \, iable$$

$$\frac{\pi}{2} f(x) = \int_{0}^{\infty} \frac{\sin \lambda}{\lambda} \cos \lambda x d\lambda$$

$$\int_{0}^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \begin{cases} \dfrac{\pi}{2} & for \, |x| \le 1 \\ 0 & for \, |x| > 1 \end{cases}$$

Answer: Putting x = 0 we get $\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}$

31. Find the Fourier sine integral representation of the function f(x) = $e^{-x}\sin x$

Solution : **[AU. NOV – 2015]**

Given f(x) = $e^{-x}\sin x$

Formula: Fourier Sine Integral

$$f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin st \,\sin sx dtds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_S(S)\sin sx\, ds \quad and \quad F_S(S) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

<u>To find</u> : $F_S(S)$

$$F_S(S) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\, dx$$

$$F_S(S) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin x \sin sx\, dx$$

$$F_S(S) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\frac{1}{2}\{\cos(1-s)x - \cos(1+s)x\}\, dx \qquad \{$$

$$\because \sin x\sin sx = \frac{1}{2}(\cos(1-s)x - \cos(1+s)x$$

$$F_S(S) = \sqrt{\frac{1}{2\pi}}\int_0^\infty e^{-x}\cos(1-s)xdx - \sqrt{\frac{1}{2\pi}}\int_0^\infty e^{-x}\cos(1+s)x\}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-x}}{1+(1-s)^2}(1.\cos(1-s)x + (1-s)\sin(1-s)x)\ \Big]_0^\infty\right] - \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-x}}{1+(1+s)^2}(1.\cos(1+s)x + (1+s)\sin(1+s)x)\ \Big]_0^\infty\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-\infty}}{1+(1-s)^2}(1.\cos(1-s)\infty + (1-s)\sin(1-s)\infty)\ \right] - \left[\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-0}}{1+(1-s)^2}(1.\cos(1-s)0 + (1-s)\sin(1-s)0)\ \right]\right]$$

$$-\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-\infty}}{1+(1+s)^2}(1.\cos(1+s)\infty + (1+s)\sin(1+s)\infty\ \right] - \left[-\frac{1}{\sqrt{2\pi}}\left[\frac{e^{-0}}{1+(1+s)^2}(1.\cos(1+s)0 + (1+s)\sin(1+s)0\ \right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \frac{1}{1+(1-s)^2}1\right] - \left[\frac{1}{\sqrt{2\pi}}\left[0 - \frac{1}{1+(1+s)^2}1\right]\right]$$

$$\because e^{-\infty} = 0\ , Cos\,\infty = 0, \sin\infty = 0, \cos 0 = 1\ and\ \ \sin 0 = 0$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{1+(1-s)^2} + \frac{1}{1+(1+s)^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-(1+(1+s)^2) + (1+(1-s)^2)}{(1+(1-s)^2)(1+(1+s)^2)}\right] \qquad = \frac{1}{\sqrt{2\pi}}\left[\frac{-4s}{(4+s^4)}\right]$$

$$F_s(S) = 2\sqrt{\frac{2}{\pi}}\left[\frac{-4s}{(4+s^4)}\right]$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_S(S)\sin sx\, ds$$

$$\therefore e^{-x}\sin x = \sqrt{\frac{2}{\pi}}\int_0^{\infty} 2\sqrt{\frac{2}{\pi}}\left[\frac{-4s}{(4+s^4)}\right]\sin sx\, ds$$

$$\therefore e^{-x}\sin x = \frac{4}{\pi}\int_0^{\infty}\left[\frac{-4s}{(4+s^4)}\right]\sin sx\, ds$$

$$\int_0^{\infty}\left[\frac{s}{(4+s^4)}\right]\sin sx\, ds = -\frac{\pi}{16}e^{-x}\sin x$$

Put S = λ

Answer: $$\int_0^{\infty}\left[\frac{\lambda}{(4+\lambda^4)}\right]\sin \lambda x\, d\lambda = -\frac{\pi}{16}e^{-x}\sin x$$

32. Find the Fourier sine and cosine transforms of a function $f(x)=e^{-x}$. Using Parseval's identity, evaluate: (1) $\int_0^{\infty}\frac{dx}{(x^2+1)^2}$ and (2) $\int_0^{\infty}\frac{x^2dx}{(x^2+1)^2}$ [AU N/D 2017]

Solution:

Given: $f(x)=e^{-x}$

Fourier cosine transform

Formula: $$F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sxdx$$

$$F_c\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sxdx$$

$$F_c\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sxdx$$

$$F_c\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sxdx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right]$$

Fourier sine transform

Formula: $$F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sxdx$$

$$F_s\left[e^{-x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+1}\right]$$

(1) By Parseval's Identity

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c[f(x)]|^2 ds$$

Let $f(x)=e^{-x}$
$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right]$

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty e^{-2x}dx$$

$$=\left[\frac{e^{-2x}}{-2}\right]_0^\infty$$

$$=\frac{1}{2}$$

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty F_c[f(x)]^2 ds$$

$$\left(\frac{1}{2}\right)=\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right]\right)^2 ds$$

$$\int_0^\infty \frac{ds}{\left(s^2+1\right)^2}=\frac{\pi}{4}$$

(2) By Parseval's Identity

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty F_s[f(x)]^2 ds \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Let $f(x)=e^{-x}$

$$\boxed{\begin{aligned} F_s[f(x)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\sin sx\,dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx \\ &= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+1}\right] \end{aligned}}$$

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty e^{-2x}dx$$

$$= \left[\frac{e^{-2x}}{-2}\right]_0^\infty$$

$$= \frac{1}{2}$$

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty F_s[f(x)]^2\,ds$$

$$\left(\frac{1}{2}\right) = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+1}\right]\right)^2 ds$$

$$\int_0^\infty \frac{s^2ds}{\left(s^2+1\right)^2} = \frac{\pi}{4}$$

Result: (1) $\displaystyle\int_0^\infty \frac{ds}{\left(s^2+1\right)^2} = \frac{\pi}{4}$

(2) $\displaystyle\int_0^\infty \frac{x^2dx}{\left(x^2+1\right)^2} = \frac{\pi}{4}$

33.If $F_s(s)$ and $F_c(s)$ denote the Fourier sine and cosine transform of a function *f(x)* respectively, then show that [AU N/D 2017]

$$F_s\left[f\left(x\right)\sin ax\right] = \frac{1}{2}\left[F_c\left(s-a\right)+F_c\left(s+a\right)\right]$$

Solution:

$$F_s\left[f\left(x\right)\sin ax\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f\left(x\right)\text{sinax}\sin sxdx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty f\left(x\right)\frac{1}{2}\left[\cos\left(s-a\right)x-\cos\left(s+a\right)x\right]\right]dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\int_0^\infty f\left(x\right)\cos\left(s-a\right)xdx-\int_0^\infty f\left(x\right)\cos\left(s+a\right)xdx\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx\right]$$

$$F_s\left[f(x)\sin ax\right] = \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$$

ANNA UNIVERSITY QUESTIONS

1. Find the Fourier Transform of $f(x)=\begin{cases} a-|x|, & for\ |x|< a \\ 0, & for\ |x|> a>0 \end{cases}$ hence deduce that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}\ ,\quad \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$$ **[AU N/D 1996] [P.NO: 21]**

2. Find the Fourier Transform of $f(x)=\begin{cases} 1-|x|, & for\ |x|<1 \\ 0, & for\ |x|>1 \end{cases}$ hence deduce that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}\ ,\ \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$$

[AU N/D 2014, N/D 2015, A/M 2016, N/D 2016] [P.NO: 23]

3. Find the Fourier transform of the function *f(x)* defined by

$$f(x)=\begin{cases} a^2-x^2 & if\ |x|<a \\ 0 & if\ |x|\geq 0 \end{cases}$$

Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt=\frac{\pi}{4}$. Using Parseval's Identity show that

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt=\frac{\pi}{15}$$ **[A.U. April, 1996, 2000, N/D 2012,2013] [P.NO: 25]**

4.Find the Fourier transform of the function *f(x)* defined by $f(x)=\begin{cases} 1-x^2 & if\ |x|<1 \\ 0 & if\ |x|\geq 1 \end{cases}$

and hence deduce that (i) $\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds=\frac{3\pi}{16}$

(ii) $\frac{\pi}{15}=\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$ **[AU A/M 2000, N/D 2013] [P.NO: 26]**

5. Find the Fourier transform of $f(x)$ is given by $f(x)=\begin{cases} 1, & for\ |x|<a \\ 0, & for\ |x|>a>0 \end{cases}$

and hence deduce that (i) $\int_0^\infty \frac{\sin t}{t}dt=\frac{\pi}{2}$ and (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$.

[AU M/J 2011, 2013, A/M 2015] [P.NO: 28]

6. Find the Fourier transform of $f(x)$ is given by $f(x)=\begin{cases} 1, & for\ |x|<2 \\ 0, & for\ |x|>2 \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin t}{t}dt$ and $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$.

[AU A/M 2017] [P.NO: 30]

7. Show that the Fourier Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$

(OR) Show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform**.**

(OR) Find the Fourier transform of $e^{\frac{-x^2}{2}}$ **[AU M/J 2013,A/M 2016] [P.NO: 32]**

8. Find the Fourier Transform of $e^{-a^2x^2}$. Hence show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform.
[AU M/J 2000, N/D 2014, A/M 2015, N/D 2016] [P.NO: 33]

9. Show that the Fourier cosine Transform of $e^{-a^2x^2}$ (OR) Show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform.
[AU N/D 1996,2012 M/J 2000, A/M 2017] [P.NO: 34]

10. Show that the Fourier cosine Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$ (OR) Show that $e^{\frac{-x^2}{2}}$ is self –reciprocal with respect to Fourier Transform.
[A.U. N/D 1996, M/J - 2000] [P.NO: 35]

11. Find the Fourier cosine transform of e^{-x^2} **[A.U. N / D 2004] [P.NO: 36]**

12. Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$ and hence find $F_C\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$
[AU NOV 2015] [P.NO: 37]

13. Find the FCT of $e^{-ax}\cos ax$ **[P.NO: 38]**

14. Find the FCT of $e^{-ax}\sin ax$ **[P.NO: 39]**

15. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$ and hence find $F_S\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$
[AU N/D 2011, N/D 2016] [P.NO: 40]

16. Find f(x), if its sine transform is $\frac{e^{-sa}}{s}$. Hence find reciprocal of $\frac{1}{s}$.
[A.U. N/D 2013] [P.NO: 41]

17. (1) Find the FCT of $\frac{1}{x^2+1}$ (2) Find the FST of $\left[\frac{x}{x^2+1}\right]$ **[P.NO: 42]**

18. Find the Fourier cosine transform of $f(x)=e^{-a^2x^2}$ and hence find the Fourier cosine transform of $e^{\frac{-x^2}{2}}$ and Fourier sine transform of $xe^{\frac{-x^2}{2}}$. **[P.NO: 43]**

19. Prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform and cosine .
[AU April, 1996, 2000, M/J 2012, Apr/May 2015] [P.NO: 44]

20. Fourier sine transform of e^{-ax} and hence find FCT of xe^{-ax}
[AU A/M 2000, M/J 2012] [P.NO: 46]

21. Find the Fourier Cosine transform of $f(x)=e^{-a^2x^2}$ and hence Fourier sine transform of $xe^{-a^2x^2}$ **[A.U. A/M, 2007, N/D 2010] [P.NO: 47]**

22. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier transforms

[A.U M/J 2006, N/D 2014] [P.NO: 48]

23. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ using transforms.

[AU N/D 2010, A/M 2017[P.NO: 49]]

24. Evaluate using transforms $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ **[A.U. M/J 2013, N/D 2013] [P.NO: 49]**

25. Evaluate using transforms $\int_0^\infty \frac{x^2dx}{(x^2+a^2)^2}$.**[A.U. N/D.2009] [P.NO: 50]**

26. State and Prove Convolution theorem. **[P.NO: 51]**
27. State and Prove Parseval's Identity. **[A.U. 2010, M/J 2012] [P.NO: 52]**
28. Find the Fourier Cosine and Sine transform of $f(x)=e^{-ax}, a>0$ and hence deduce the inversion formula (OR) Find the Fourier Cosine and Sine transform of $f(x)=e^{-ax}, a>0, x\ge 0$. Hence deduce integrals $\int_0^\infty \frac{\cos sx}{s^2+a^2}ds$ and $\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$ **[AU N/D 2012 A/M 2016][P.NO: 52]**
29. Verify convolution theorem for $f(x)=g(x)=e^{-x^2}$

[A.U. M/J 2013,NOV 2015] [P.NO: 54]

30. Express the function $f(x)=\begin{cases}1 & for|x|\le 1\\ 0 & for|x|>1\end{cases}$ as a Fourier Integral. Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda$ and find the value of $\int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda$ **[AU April, 2001] [P.NO: 56]**
31. Find the Fourier sine integral representation of the function f(x) = $e^{-x}\sin x$

[AU. NOV – 2015][P.NO: 58]

32. Find the Fourier sine and cosine transforms of a function $f(x)=e^{-x}$. Using Parseval's identity, evaluate: (1) $\int_0^\infty \frac{dx}{(x^2+1)^2}$ and (2) $\int_0^\infty \frac{x^2dx}{(x^2+1)^2}$

[AU N/D 2017] [P.NO: 59]

33. If $F_s(s)$ and $F_c(s)$ denote the Fourier sine and cosine transform of a function *f(x)* respectively, then show that

$$F_s\left[f(x)\sin ax\right]=\frac{1}{2}\left[F_c(s-a)+F_c(s+a)\right]$$ **[AU N/D 2017] [P.NO:61]**

Important Questions
PART-A

1. State Fourier integral theorem
[A.U. April 1996, April/May 2005, May-2016]
2. Show that $f(x)=1, 0<x<\infty$ cannot be represented by a Fourier integral.
[A.U. April/May 2003]
3. Define Fourier Transform pair. **[A U March, 1996]**
4. What is the Fourier cosine transform of a function. (or)
Write down the Fourier cosine transform pair of formulae. **[AU O/N 1996]**
5. Find the Fourier cosine transform of $e^{-ax}, a>0$ **[A U, April, 2001]**
6. Find the Fourier cosine transform of e^{-3x}
7. Find the Fourier sine transform of e^{-3x} **[A U, Nov / Dec 1996, M/J 2013]**
8. Find the Fourier Sine transform of $\frac{1}{x}$ **[AU A/M 2015, N/D 2016, A/M 2017]**
9. Define Fourier sine transform and its inversion formula. **[AU A/M 2004]**
10. Find the Fourier sine transform of f(x) = e^{-x} and hence deduce that
$$\int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-m}$$ **[A U, March, 1998, 1999 & 2000]**
11. Find $F_c\left[xe^{-ax}\right]$ and $F_s\left[xe^{-ax}\right]$
12. Show that the Fourier sine transforms of $xe^{-x^2/2}$ is self reciprocal.
[A U, March, 1996]
13. If Fourier transform of f(x) is F(s), prove that the Fourier transform of f(x) $\cos ax$ **is** $\frac{1}{2}\left[F(s-a)+F(s+a)\right]$ **[AU April, 2001, Nov/Dec 2014]** **(or)**
State and Prove Modulation Theorem.
14. Prove that $F_C\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_C(s+a)+F_C(s-a)\right]$ where F_c denotes the Fourier cosine transform f(x). **[AU A/M 2001]**
15. If F(s) is the complex Fourier transform of f(x) then find $F\left[\delta(x-a)\right]$
[AU A/M 2000]
16. Given that $e^{-x^2/2}$ is self reciprocal under Fourier cosine transform, find (i) Fourier sine transform of $xe^{-x^2/2}$ and (ii) Fourier cosine transform of $x^2e^{-x^2/2}$ **[A U, DEC 1996]**
17. If $F_c(s)$ is the Fourier cosine transform of *f(x)*, Prove that the Fourier cosine transform of *f(ax)* is $\frac{1}{a}F_c\left[\frac{s}{a}\right]$
18. If F(s) is the Fourier transform of *f(x)*, then find the Fourier transform of *f(x-a)*. (or) State and prove shifting theorem.
(or) If the Fourier Transform of f(x) is F[f(x)] = F(s), then show that
F[f(x-a)] $=e^{ias}$F(S) **[AU N/D 2013, A/M 2015, A/M 2017]**

19. If $F_s(s)$ is the Fourier sine transform of *f(x)*,

show that $$F_s\left[f(x)\cos ax\right]=\frac{1}{2}\left[F_s(s+a)+F_s(s-a)\right]$$

20. State the convolution theorem for Fourier transforms.

(or)

State the Faltung theorem. **[A.U. April / May 2003, May 200 PT]**

21. Find the Fourier transform of $f(x)=\begin{cases}1 & in\ |x|<a\\ 0 & in\ |x|>a\end{cases}$

22. Find the Fourier cosine transform of $f(x)=\begin{cases}\cos x\ if\ 0<x<a\\ 0\quad if\quad x\geq a\end{cases}$

23. Find the Fourier Cosine transform of $e^{-2x}+3e^{-x}$

24. Find Fourier Cosine transform of e^{-x}.

25. If Fourier transform of $f(x)=F(s)$ then what is Fourier transform of $f(ax)$

(OR) If F(s) is the fourier transforms of f(x), Prove that $F\left[f(ax)\right]=\frac{1}{a}F\left(\frac{s}{a}\right)$ (or)

State and Prove a Change of scale property.

[AU M/J 2013, N/D 2015, N/D 2016]

26. If F denotes the Fourier transform operator then show that

$$F\left\{x^n f(x)\right\}=(-i)^n\frac{d^n}{ds^n}F\left\{s\right\}$$

27. If F(s) is the Fourier transform of $f(x)$, show that the Fourier transform of $e^{iax}f(x)\ is\ F(s+a)$. **[AU Nov/Dec 2014]**

28. Find $F\left[\frac{d^n}{dx^n}f(x)\right]$

29. State Parseval's Identity for Fourier transform.

30. Find the Fourier transform of $f(x)=\begin{cases}e^{ikx} & ,\ a<x<b\\ 0\, x<a\ and\ x>b\end{cases}$

31. State the Fourier transform of the derivatives of a function. (OR) Find the Fourier transform of a derivative of the function f(x) if f(x) $\to 0$ as x $\to \pm\infty$

[AU N/D 2005, M/J -2016]

32. Find the Fourier sine transform of e^{-x} **[AU M /J 2006]**

33. Find the Fourier Cosine transform of of $f(x)$ defined as **[AU N /D 2006]**

$$f(x)=\begin{cases}x & for\quad 0<x<1\\ 2-x & for\quad 1<x<2\\ 0 & for\quad x>2\end{cases}$$

34. Find the Fourier sine transform of $f(x)=e^{-ax}$, $a>0$ **[AU N/D 2007]**

35. State inverse theorem for complex Fourier transform.

36. Find the function $f(x)$ whose sine transform is e^{-as}. **[AU M/J 2010]**

37. Find the Fourier cosine transform of e^{-2x} **[AU N/D 2010]**

38. Prove that $F_S[xf(x)] = -\frac{d}{ds}[F_C(s)]$ **[AU A/M 2011]**

39. Define the Fourier sine and Cosine transform of $f(x)$.

40. Find the Fourier transform of $e^{-\alpha|x|}, \alpha > 0$. **[AU N/D 2012]**

41. Find the Fourier sine transform of $\frac{e^{-ax} - e^{-bx}}{x}$

42. Find the Fourier sine transform of $\frac{x}{x^2 + a^2}$

43. Find the Fourier Cosine transform of $5e^{-2x} + 2e^{-5x}$

44. If $f(x) = \begin{cases} \sin x & 0 \le x \le a \\ 0 & x > a \end{cases}$, Find the Sine transform.

45. Find the function $f(x)$ whose sine transform is $\frac{e^{-as}}{s}$.

46. Define self reciprocal with respect to Fourier Transform. **[AU N/D 2013]**

47. State Convolution theorem. [AU N/D 2012]

48. Evaluate $\int_0^\infty \frac{s^2 ds}{(s^2 + a^2)(s^2 + b^2)}$ using Fourier transforms **[A.U N/D 2015]**

<u>PART-B</u>

1. Find the Fourier Transform of $f(x) = \begin{cases} a - |x|, & for\ |x| < a \\ 0, & for\ |x| > a > 0 \end{cases}$ hence deduce that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}, \quad \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$ **[AU Nov / Dec, 1996]**

2. Find the Fourier Transform of $f(x) = \begin{cases} 1 - |x|, & for\ |x| < 1 \\ 0, & for\ |x| > 1 \end{cases}$ hence deduce that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}, \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$ **[AU N/D 2014, N/D 2015, A/M 2016, N/D 2016]**

3. Find the Fourier transform of the function *f(x)* defined by

$$f(x) = \begin{cases} a^2 - x^2 & if\,|x| < a \\ 0 & if\,|x| \ge 0 \end{cases}$$

Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3} dt = \frac{\pi}{4}$. Using Parseval's Identity show that

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$$ **[A.U. April, 1996, 2000, N/D 2012,2013]**

4.Find the Fourier transform of the function *f(x)* defined by $f(x)=\begin{cases}1-x^2 & if\,|x|<1\\ 0 & if\,|x|\geq 1\end{cases}$

and hence deduce that (i) $\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{16}$

(ii) $\frac{\pi}{15}=\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$ **[AU A/M 2000, N/D 2013]**

5. Find the Fourier transform of $f(x)$ is given by $f(x)=\begin{cases}1, & for\ |x|<a\\ 0, & for\ |x|>a>0\end{cases}$

and hence deduce that $(i)\ \int_0^\infty \frac{\sin t}{t}dt=\frac{\pi}{2}$ and $(ii)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$.

[AU M/J 2011, 2013, A/M 2015]

6. Find the Fourier transform of $f(x)$ is given by $f(x)=\begin{cases}1, & for\ |x|<2\\ 0, & for\ |x|>2\end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin t}{t}dt$ and $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$.

[AU A/M 2017]

7. (a)Show that the Fourier Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$

(OR) Show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform**.**

(OR) Find the Fourier transform of $e^{\frac{-x^2}{2}}$ **[AU M/J 2013,MAY-2016]**

(b)Find the Fourier Transform of $e^{-a^2x^2}$. Hence show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform.

[AU M/J 2000, N/D 2014, A/M 2015, N/D 2016]

8.(a).Show that the Fourier cosine Transform of $e^{-a^2x^2}$ (OR) Show that $e^{\frac{-x^2}{2}}$ is self – reciprocal with respect to Fourier Transform.

[AU N/D 1996,2012 M/J 2000, A/M 2017]

(b) Show that the Fourier cosine Transform of $e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$ (OR) Show that $e^{\frac{-x^2}{2}}$ is self –reciprocal with respect to Fourier Transform.

[A.U. N/D 1996, M/J - 2000]

9(a) Find the Fourier cosine transform of e^{-x^2} **[A.U. N / D 2004]**

(b) Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$ and hence find $F_C\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$

[AU NOV 2015]

10(a). Find the FCT of $e^{-ax}\cos ax$

(b) Find the FCT of $e^{-ax}\sin ax$

11(a) Find the Fourier sine transform of $\frac{e^{-ax}}{x}$ and hence find $F_S\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$

[AU N/D 2011, N/D 2016]

(b) Find f(x), if its sine transform is $\frac{e^{-sa}}{s}$. Hence find reciprocal of $\frac{1}{s}$.

[A.U. N/D 2013]

12. (a)(1) Find the FCT of $\frac{1}{x^2+1}$ (2) Find the FST of $\left[\frac{x}{x^2+1}\right]$

(b) Find the Fourier cosine transform of $f(x)=e^{-a^2x^2}$ and hence find the Fourier cosine transform of $e^{\frac{-x^2}{2}}$ and Fourier sine transform of $xe^{\frac{-x^2}{2}}$.

13(a). Prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine transform and cosine .

[AU April, 1996, 2000, M/J 2012, Apr/May 2015]

(b) Fourier sine transform of e^{-ax} and hence find FCT of xe^{-ax}

[AU A/M 2000, M/J 2012]

14(a). Find the Fourier Cosine transform of $f(x)=e^{-a^2x^2}$ and hence Fourier sine transform of $xe^{-a^2x^2}$ **[A.U. A/M, 2007, N/D 2010]**

(b). Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using Fourier transforms

[A.U M/J 2006, N/D 2014]

15(a). Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ using transforms**. [AU N/D 2010, A/M 2017]**

(b). Evaluate using transforms $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$. **[A.U. M/J 2013, N/D 2013]**

16(a). Evaluate using transforms $\int_0^\infty \frac{x^2dx}{(x^2+a^2)^2}$. **[A.U. N/D.2009]**

(b) State and Prove Convolution theorem.

17(a). State and Prove Parseval's Identity. **[A.U. 2010, M/J 2012]**

(b) Find the Fourier Cosine and Sine transform of $f(x)=e^{-ax}, a>0$ and hence deduce the inversion formula (OR) Find the Fourier Cosine and Sine transform of $f(x)=e^{-ax}, a>0, x\ge 0$. Hence deduce integrals $\int_0^\infty \frac{\cos sx}{s^2+a^2}ds$ and $\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$ **[AU N/D 2012 A/M 2016]**

18(a).Verify convolution theorem for $f(x) = g(x) = e^{-x^2}$ **[A.U. M/J 2013,NOV 2015]**

(b). Express the function $f(x) = \begin{cases} 1 & for|x| \le 1 \\ 0 & for|x| > 1 \end{cases}$ as a Fourier Integral. Hence evaluate

$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$ and find the value of $\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda$ **[A U, April, 2001]**

19.Find the Fourier sine integral representation of the function f(x) = $e^{-x} \sin x$

[AU. NOV – 2015]

MAILAM ENGINEERING COLLEGE

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II Year B.E (Civil, EEE & Mech.)

DEPARTMENT OF MATHEMATICS

SUBJECT NAME : MA8353 - TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS

UNIT – V (Z - TRANSFORMS)

Syllabus: Z-Transform-Elementary properties-Inverse Z-Transform(using partial fraction and residues) – Convolution theorem-Formation of difference equations- Solution of difference equations using Z-Transforms.

No. of pages : 63+1+27

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	49	17	17	6
Part - B	4	20	3,46	2,16
	23	39	7	22
	39	51	13	28
	44	54	43	54

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PRINCIPAL

UNIT –V
Z-TRANSFORM AND DIFFERENCE EQUATIONS

PART -A

1. Define Z - transform **[AU A/M 2009, 2007]**

Solution:

Let {f (n)} be a sequence defined for n =0, ±1, ± 2……, then Z- transform is defined as

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n} \quad , (z \to \text{a complex number})$$

This is called two sided or bilateral Z- transform

Definition:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} \quad (z \to \text{a complex number})$$
$$= F(Z)$$

This is called one sided Z - transform

Definition: Z - transform for discrete values of 't'

If f (t) is defined for discrete value of't' where t = nT, n=0, 1, 2, 3,…T being the sampling , then

$$Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$
$$= F(Z)$$

2. Prove that $Z\left[a^n\right] = \dfrac{z}{z-a}$

Solution: **[AU M/J 2005, 2009, 2017]**

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\{(a^n)\} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} \qquad \left[\because (1-x)^{-1} = 1 + x + x^2 + \dots \quad Here\ x = \frac{a}{z}\right]$$

$$= \frac{1}{1 - \frac{a}{z}}$$

$$= \frac{1}{\frac{z-a}{z}}$$

$$= \frac{z}{z-a}$$

$$Z\left[a^n\right] = \frac{z}{z-a}$$

3. Find $Z(n)$ (or) Find the Z-transform of {n} [AU N/D 2010, A/M 2018, 2000, 2013]

Solution:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z(n) = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{n}{z^n}$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \ldots\right]$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)^{2} \qquad \because \begin{cases} (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots \text{ if } |x| < 1 \\ Here \quad x = \frac{1}{z} \quad i.e., \left|\frac{1}{z}\right| > 1 \end{cases}$$

$$= \frac{1}{z} \cdot \frac{z^2}{(z-1)^2}$$

$$\therefore Z(n) = \frac{z}{(z-1)^2}$$

4. Find $Z\{f(n)\}$ **if** $f(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$ **[AU N/D 2008, M/J 2007]**

Solution:

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} n z^{-n} \qquad [\because f(n) = n, n \geq 0]$$

$$= \frac{z}{(z-1)^2}$$

5. Find $Z\left[\frac{a^n}{n!}\right]$ in Z- transform [AU N/D -2005]

Solution:

$$Z\{f(n)\}=\sum_{n=-\infty}^{\infty} f(n)z^{-n}$$

$$Z\left[\frac{a^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{a^n}{n!} \qquad \left[f(n)=\begin{cases}a^n, & n\geq 0\\ 0, & otherwise\end{cases}\right]$$

$$=\sum_{n=0}^{\infty}\left(\frac{(az^{-1})^n}{n!}\right)$$

$$=1+\frac{az^{-1}}{1!}+\frac{(az^{-1})^2}{2!}+.... \qquad \begin{cases}\because e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+...\\ Here\, x=az^{-1}\end{cases}$$

$$=e^{az^{-1}}$$

$$=e^{\frac{a}{z}}$$

6. Find $Z[a^{|n|}]$ **[AU N/D 2010, A/M 2002, 2009]**

Solution:

$$Z\{f(n)\}=\sum_{n=-\infty}^{\infty} f(n)z^{-n}$$

$$=\sum_{n=-\infty}^{\infty} a^{|n|}z^{-n}$$

$$=\sum_{n=-\infty}^{-1} a^{(-n)}\,z^{-n}+\sum_{n=0}^{\infty}a^n\,z^{-n}$$

$$=\sum_{n=-\infty}^{-1}(a\,z)^{-n}+\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n \qquad \begin{cases}formula: |n|=-n(-\infty,-1)\\ |n|=n(0,\infty)\end{cases}$$

$$=\left[az+(az)^2+...\right]+\left[1+\frac{a}{z}+\frac{a^2}{z^2}+....\right]$$

$$=az\left(1+az+(az)^2+...\right)+\left[1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+....\right]$$

$$=az(1-az)^{-1}+\left(1-\frac{a}{z}\right)^{-1} \qquad \begin{cases}\because (1-x)^{-1}=1+x+x^2+...\\ Here\,x=az \quad and\, x=\frac{a}{z}\end{cases}$$

$$=az\frac{1}{1-az}+\frac{1}{1-\frac{a}{z}}$$

$$=\frac{az}{1-az}+\frac{1}{\frac{z-a}{z}}$$

$$=\frac{az}{1-az}+\frac{z}{z-a}$$

$$=\frac{az(z-a)+(1-az)z}{(1-az)(z-a)}$$

$$= \frac{az^2 - a^2z + z - az^2}{(1-az)(z-a)}$$

$$= \frac{z - a^2z}{(1-az)(z-a)}$$

7. Find $Z(-1)^n$ **[AU M/J 2005, 2010]**

Solution:

$$Z\left[a^n\right] = \frac{1}{z-a}$$

$$\therefore z\left[(-1)^n\right] = \frac{1}{z-(-1)} \qquad \left[Here\, a = -1\right]$$

$$= \frac{1}{z+1}$$

8. Find $Z\left[e^{-an}\right]$ **[AU M/J 2007, 2008]**

Solution:

$$z\left[(a)^n\right] = \frac{1}{z-a}$$

$$\because z\left[(e^{-a})^n\right] = \frac{1}{z-(e^{-a})}$$

9. Find Z[t]

Solution:

$$Z\{f\ (n)\} = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$Z\left[t\right] = \sum_{n=-\infty}^{\infty} nT\, z^{-n} \qquad \left[\begin{array}{l} f(t) = t \\ \text{Re } place\, t \rightarrow nT \end{array}\right]$$

$$= T\sum_{n=0}^{\infty} nz^{-n}$$

$$= \text{T } Z[\text{n}]$$

$$= T\frac{z}{(z-1)^2} \qquad \left[\because z(n) = \frac{z}{(z-1)^2}\right]$$

$$\therefore \qquad Z\left[t\right] = \frac{Tz}{(z-1)^2}$$

10. Find $Z[a^{n-1}]$

Solution:

$$Z\left[a^{n-1}\right] = Z\left[a^{-1}.a^n\right]$$

$$= a^{-1}Z\left[a^n\right]$$

$$= a^{-1}.\frac{Z}{Z-a}$$

$$= \frac{1}{a}.\frac{Z}{Z-a}$$

11. Find $Z\left[\frac{1}{n(n+1)}\right]$

Solution: **[AU M/J 2016, N/D 2016]**

$$\text{Let } \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \qquad \ldots(1)$$

$$1 = A(n+1) + B(n)$$

Put $n = 0$, $A = 1$

$n = -1$, $B = -1$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{(n+1)}\right]$$

$$= \log\left[\frac{z}{z-1}\right] - z\log\left[\frac{z}{z-1}\right]$$

$$= (1-z)\log\left[\frac{z}{z-1}\right]$$

12. Find $Z\left[\sin\frac{n\pi}{2}\right]$ **and** $Z\left[\cos\frac{n\pi}{2}\right]$ **[AU N/D 2009, M/J 2007, '16]**

Solution:

$$Z[\sin n\theta] = \frac{(z\sin\theta)}{(z^2 - 2z\cos\theta + 1)} \qquad Z[\cos n\theta] = \frac{z(z-\cos\theta)}{(z^2 - 2z\cos\theta + 1)}$$

Put $\theta = \frac{\pi}{2}$

$$Z\left[\sin\frac{n\pi}{2}\right] = \frac{(z\sin\frac{\pi}{2})}{(z^2 - 2z\cos\frac{\pi}{2} + 1)} \qquad Z\left[\cos\frac{n\pi}{2}\right] = \frac{z\left(z-\cos\frac{\pi}{2}\right)}{(z^2 - 2z\cos\frac{\pi}{2} + 1)}$$

$$= \frac{z}{z^2+1} \qquad = \frac{z^2}{z^2+1}$$

13. Find $z[e^{at+b}]$

Solution:

$$Z[e^{at}] = \frac{z}{z-e^{at}}$$

$$z[e^{at+b}] = Z[e^b.e^{at}]$$

$$= e^b \; Z[e^{at}] = e^b \frac{z}{z-e^{at}}$$

14. Find Z $[e^{3t+2}]$ **[AU N/D 2008 , M/J 2007]**

Solution:

$$Z[e^{at}] = \frac{z}{z-e^{at}}$$

$$= Z[e^2 . e^{3t}]$$

$$= e^2 \frac{z}{z - e^{3t}}$$

15. State and Prove first shifting theorem

If $Z[f(t)] = F(Z)$ **, then** $Z[e^{-at} f(t)] = F[ze^{at}]$

Proof:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$Z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (ze^{aT})^{-n}$$

$$= F[ze^{aT}]$$

16. Find $Z[e^{-iat}]$

Solution:

$$Z[e^{-iat}] = Z[e^{-ait} .1]$$

$$= \{z[1]\}_{z \to ze^{iat}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{iat}}$$

$$= \frac{ze^{iat}}{ze^{iat} - 1}$$

17. Prove that $Z[n f(n)] = -Z\frac{d}{dz}\{F(Z)\}$ [AU A/M 2018]

Proof:

$$wkt, \quad F(Z) = Z[f(n)]$$

$$F(Z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\frac{d}{dz} F(Z) = \frac{d}{dz} \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n) \frac{d}{dz}\left(z^{-n}\right)$$

$$= \sum_{n=0}^{\infty} f(n)(-n) z^{-n-1}$$

$$= \sum_{n=0}^{\infty} - n. f(n) z^{-n} . \frac{1}{z}$$

$$Z \frac{d}{dz}[F(z)] = -\sum_{n=0}^{\infty} n f(n) z^{-n}$$

$$= -z[nf(n)]$$

$$Z[nf(n)] = -z \frac{d}{dz} F(z)$$

18. Find the Z transform of $^{n}c_{k}$

Solution:

$$z\{nC_k\} = \sum_{k=0}^{n} nC_k z^{-k}$$
$$= 1 + nC_1 z^{-1} + nC_2 z^{-2} + ... + nC_n z^{-n}$$

This is expansion of binomial theorem

$$= (1+z^{-1})^n$$

19. Find $Z(n^2)$ *and* $Z(n^3)$

[A.U M/J 2014, N/D 2016]

Solution:

$$Z(n^2) = Z(n\times n)$$
$$= -z\frac{d}{dz}[Z(n)]$$
$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$
$$= -z\left\{\frac{(z-1)^2 - z.2(z-1)}{(z-1)^4}\right\}$$
$$= -z\left\{\frac{z-1-2z}{(z-1)^3}\right\}$$
$$= \frac{z(z+1)}{(z-1)^3}$$
$$= \frac{z^2+z}{(z-1)^3}$$

$$Z(n^3) = Z(n\times n^2)$$
$$= -z\frac{d}{dz}\left[Z(n^2)\right]$$
$$= -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$
$$= -z\left\{\frac{(z-1)^3(2z+1) - (z^2+z)3(z-1)^2}{(z-1)^6}\right\}$$
$$= -z\left\{\frac{(z-1)(2z+1) - (z^2+z)3}{(z-1)^4}\right\}$$
$$= -z\left\{\frac{2z^2+z-2z-1-3z^2-3z}{(z-1)^4}\right\}$$
$$= -z\left\{\frac{-z^2-4z-1}{(z-1)^4}\right\} = z\left\{\frac{z^2+4z+1}{(z-1)^4}\right\}$$

20. Find $Z[an^2+bn+c]$

Solution:

$$Z\left[an^2+bn+c\right] = az(n^2) + bz(n) + cz(1)$$
$$= a\frac{z^2+z}{(z-1)^3} + b\frac{z}{(z-1)^2} + c\frac{z}{z-1}$$

21. Find $Z[n(n-1)]$

[May 2009, Apr2007]

Solution:

$$Z[n(n-1)] = Z\left[n^2 - n\right]$$
$$= z\left[n^2\right] - z[n]$$
$$= \frac{z^2+z}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z^2 + z - z(z-1)}{(z-1)^3}$$

$$= \frac{1}{(z-1)^3}\left[z^2 + z - z^2 + z\right]$$

$$= \frac{2z}{(z-1)^3}$$

22. Prove that the Damping Rule

$$Z\left[a^n f(n)\right] = F\left(\frac{z}{a}\right), \quad F(z) = Z\left[f(n)\right]$$

Proof:

$$Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$\therefore \quad Z\left[a^n f(n)\right] = \sum_{n=0}^{\infty} a^n f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n}$$

$$Z\left[a^n f(n)\right] = F\left(\frac{z}{a}\right)$$

23. Find $Z[a^n \sin n\theta]$ **[AU A/M 2005, 2007]**

Solution:

$$Z\left[a^n f(n)\right] = F\left(\frac{z}{a}\right)$$

$$Z[\left[a^n \sin n\theta\right] = Z\left[\sin n\theta\right]_{z \to \frac{z}{a}}$$

$$= \left[\frac{z \sin n\theta}{z^2 - 2z\cos\theta + 1}\right]_{z \to \frac{z}{a}}$$

$$= \frac{\frac{z}{a}\sin n\theta}{\frac{z^2}{a^2} - \frac{2z}{a}\cos\theta + 1}$$

$$Z\left[a^n \sin n\theta\right] = \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

24. Find $Z[na^n]$

Solution:

$$Z\left(na^n\right) = z\left(a^n . n\right)$$

$$= \left\{Z(n)\right\}_{z \to \frac{z}{a}}$$

$$= \left\{\frac{z}{(z-1)^2}\right\}_{z \to \frac{z}{a}}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}$$

$$= \frac{\frac{z}{a}}{\frac{(z-a)^2}{a^2}}$$

$$= \frac{z}{a} \times \frac{a^2}{(z-a)^2}$$

$$= \frac{az}{(z-a)^2}$$

$$Z[na^n] = \frac{az}{(z-a)^2}$$

25. Find $Z[a^n t]$

Solution: **[AU N/D 2010, A/M 2009]**

$$z[a^n t] = [z(t)]_{z \to \frac{z}{a}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to \frac{z}{a}}$$

$$= \frac{T\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}$$

$$= \frac{Taz}{(z-a)^2}$$

26. Find $Z[e^{-t}\sin 2t]$

Solution:

$$Z[e^{-at} f(t)] = [F(Z)]_{z \to ze^{aT}}$$

$$Z[e^{-t}\sin 2t] = [Z(\sin 2t)]_{z \to ze^{T}}$$

$$= \left[\frac{z\sin 2T}{z^2 - 2z\cos 2T + 1}\right]_{z \to ze^{T}} \qquad \left[\begin{array}{l} formula: \\ z[\sin at] = \dfrac{z\sin aT}{z^2 - 2z\cos aT + 1} \end{array}\right]$$

$$= \frac{ze^T \sin 2T}{z^2 e^{2T} - 2ze^T \cos 2T + 1}$$

27. Find $Z[a^{n-1}]$

Solution:

We know that,

$$Z[a^n] = \frac{z}{z-a}$$

$$\therefore Z\left[a^{n-1}\right] = z^{-1}.Z\left[a^{n}\right], \quad \text{by shift property}$$

$$= \frac{1}{z}.\frac{z}{z-a}$$

$$Z\left[a^{n-1}\right] = \frac{1}{z-a}$$

28. Prove that $Z[\, f(n+1)] = z\,F(z) - z\,f(0)$

Proof: **[AU N/D 2005]**

$$Z\left[f(n+1)\right] = \sum_{n=0}^{\infty} f(n+1)z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n+1)z.z^{-1}.z^{-n}$$

$$= z\sum_{n=0}^{\infty} f(n+1)z^{-(n+1)}$$

Put n+1=m and we get $\begin{cases} n=0 \to m=1 \\ n=\infty \to m=\infty \end{cases}$

$$Z\left[f(n+1)\right] = z\sum_{m=1}^{\infty} f(m)z^{-m} + zf(0) - zf(0)$$

$$= z\left[\sum_{m=0}^{\infty} f(m)z^{-m} - f(0)\right]$$

$$= zF(z) - zf(0)$$

29. Define Unit step sequence.

Solution:

A discrete unit step function is defined as

$$U(K) = \begin{cases} 1, K \geq 0 \\ 0, K < 0 \end{cases}$$

30. Find Z[cos(t+T)] **[AU N/D 2009, M/J 2004,2008]**

Solution:

$$z\left[f(t+T)\right] = ZF(z) - Zf(0)$$

$$Here \;\; f(t) = \cos(t) \quad , f(0) = \cos 0 = 1$$

$$Z\left[\cos(t+T)\right] = z.Z[\cos t] - zf(0)$$

$$= z\frac{z(z-\cos T)}{z^2 - 2z\cos T + 1} - z$$

$$= \frac{z^2(z-\cos T)}{z^2 - 2z\cos T + 1} - z$$

$$= \frac{z^3 - z^2\cos T - z(z^2 - 2z\cos T + 1)}{z^2 - 2z\cos T + 1}$$

$$= \frac{z^2\cos T - z}{z^2 - 2z\cos T + 1}$$

31. Define unit impulse sequence **[AU M/J 2004, 2005]**

Solution:

A discrete unit step function is defined as

$$\delta(K)=\begin{cases}1, K=0\\ 0, K\neq 0\end{cases}$$

Now δ (n-k) is defined by

$$\delta(n-k)=\begin{cases}1, n=k\\ 0, n\neq k\end{cases}$$

32. Find $Z[\gamma(n-k)]$ **[AU M/J 2005, 2010]**

Solution:

$$Z[\gamma(n-k)]=\sum_{n=0}^{\infty}\gamma(n-k)z^{-n}$$

$$=\gamma(0-k)z^{-0}+\gamma(1-k)z^{-1}+\gamma(2-k)z^{-2}+....$$

$$\gamma(n-k)=\begin{cases}1, n=k\\ 0, n\neq k\end{cases}$$

$$\therefore z[\gamma(n-1)]=z^{-k}=\frac{1}{z^k}$$

Inparticular,

$$Z[\gamma(n-1)]=\frac{1}{z}$$

33. Find $Z[2^n\gamma(n-2)]$

Solution:

$$Z[2^n\gamma(n-2)]=\{Z[\gamma(n-2)]\}_{z\to\frac{z}{2}}$$

$$=\left(\frac{1}{z^2}\right)_{z\to\frac{z}{2}}$$

$$=\frac{1}{\left(\frac{z}{2}\right)^2}$$

$$=\frac{4}{z^2}$$

$$\left[\begin{array}{l}Formula:\\ Z[\gamma(n-k)=\dfrac{1}{z^k}\end{array}\right]$$

$$Z[2^n\gamma(n-2)]=\frac{4}{z^2}$$

34. Prove the Initial value theorem **[N/D 2007, M/J 2004]**

Statement:

If $Z[f(n)]=F(z)$**, then** $\lim\limits_{Z\to\infty}F(z)=f(0)$

Proof:

$$z[f(n)]=\sum_{n=0}^{\infty}f(n)z^{-n}$$

$$=f(0)+f(1)z^{-1}+f(2)z^{-2}+....$$

$$F(Z) = f(0) + \frac{f(1)}{Z} + \frac{f(2)}{Z^2} + ...$$

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + ...\right]$$

$$= f(0)$$

$$\left[\begin{array}{l} \text{Result}: \\ \lim_{z\to\infty}\frac{1}{z} = 0 \end{array}\right]$$

35. If $F(Z) = \dfrac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$ **Find** f(0)

Solution:

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$$

$$= \lim_{z\to\infty} \frac{(z^2 - z\cos aT)}{z^2 - 2z\cos aT + 1}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}(z^2 - z\cos aT)}{\frac{d}{dz}(z^2 - 2z\cos aT + 1)}$$

$$= \lim_{z\to\infty} \frac{(2z - \cos aT)}{2z - 2\cos aT}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}(2z - \cos aT)}{\frac{d}{dz}(2z - 2\cos aT)}$$

$$= \lim_{z\to\infty} \frac{2}{2}$$

$$= 1$$

36. Find the final value of the function $F(z) = \dfrac{1+z^{-1}}{1-0.25z^{-2}}$ **[M/J 2007, 2009]**

Solution:

By the final value theorem,

$$f(\infty) = \lim_{t\to\infty} f(t)$$

$$= \lim_{z\to 1}(z-1)F(Z)$$

$$= \lim_{z\to 1}(z-1).\frac{1+z^{-1}}{1-0.25z^{-2}}$$

$$= \lim_{z\to 1}(z-1)\frac{1+\frac{1}{z}}{1-\frac{0.25}{z^2}}$$

$$= \frac{(1-1)(1+1)}{1-0.25}$$

$$= 0$$

37. Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$, Using residue method. **[AU N/D 2007, 2008, 2016]**

Solution:

$$Let \quad F(Z) = \frac{z^2}{(z-a)(z-b)} \quad , \quad z^{-1}[F(z)] = f(n)$$

$$\therefore z^{n-1}F(z) = \frac{z^{n+1}}{(z-a)(z-b)}$$

The poles are $z = a,\ z = b$

$$\mathrm{Res}\{z^{n-1}F(z)\}_{z=a} = \lim_{z\to a}(z-a)\frac{z^{n+1}}{(z-a)(z-b)}$$

$$= \frac{a^{n+1}}{a-b}$$

$$\mathrm{Res}\{z^{n-1}F(Z)\}_{Z=b} = \lim_{z\to b}(z-b)\frac{z^{n+1}}{(z-a)(z-b)}$$

$$= \frac{b^{n+1}}{b-a}$$

$f(n) =$ {sum of the residue of z^{n-1} F (z) at its poles}

$$= \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right]$$

$$\therefore z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right]$$

38. Form a difference equation by eliminating the arbitrary constant 'A' from $y_n = A.3^n$ **[AU M/J 2007, 2009, N/D 2010]**

Solution:

$$y_n = A.3^n$$

$$y_{n+1} = A.3^{n+1}$$

$$= 3A.3^n$$

$$= 3y_n$$

$$y_{n+1} - 3y_n = 0$$

39. Define convolution of two sequences. **[AU M/J 2003, 2005]**

Solution:

The convolution of two sequence {f(n)} and {g(n)} is defined as

$$\left[f(n) * g(n)\right] = \sum_{r=0}^{n} f(r)g(n-r) \qquad \text{[For right sided sequences]}$$

(or)

$$\left[f(n)*g(n)\right]=\sum_{r=-\infty}^{\infty} f(r)g(n-r)$$ [For two sided and bilateral sequence]

The convolution of two function f(t) and g(t) is defined as

$$\left[f(t)*g(t)\right]=\sum_{r=-\infty}^{\infty} f(rT)g(n-r)T$$ Where T is the sampling period

40. State Convolution theorem on Z-transforms **[A.U M/J 2015, N/D 2016]**

Solution:

$1.z[f(n)*g(n)]=F(Z).G(Z)$

$2.Z[f(t)*g(t)]=F(Z).G(Z)$

41. Find the Z- transform of $f(n)*g(n)$ **if** $f(n) = 2^n u(n)$ **and** $g(n) = 2^n u(n)$

Solution:

$$Z\left[f(n)\right]=Z\left[2^n u(n)\right]$$
$$=\frac{z}{z-2}$$
$$=F(Z)$$
$$Z\left[g(n)\right]=Z\left[2^n u(n)\right]$$
$$=\frac{z}{z-2}$$
$$=G(z)$$ by def. of u(n)

$$\therefore Z\left[f(n)*g(n)\right]=\left[F(Z).G(Z)\right]$$
$$=\frac{z}{z-2}.\frac{z}{z-2}$$
$$=\left(\frac{z}{z-2}\right)^2$$

42. State initial and final value theorem in Z-transform

Solution: **[M/J 2003, 2007, 2017, N/D 2014, 2015]**

Initial value theorem

If $z\left|f(n)\right|=F(z)$, then $\lim_{Z\to\infty} F(z)=f(0)$

Final value theorem

If z [f (t)] =F (Z) then $\lim_{t\to\infty} f(t)=\lim_{z\to 1}(z-1)F(Z)$

43. Find the Z-Transform of 1/n **[AU N/D 2013, N/D 2017]**

Solution:

$$Z[x(n)]=\sum_{n=0}^{\infty} x(n)z^{-n}$$
$$Z\left[\frac{1}{n}\right]=\sum_{n=1}^{\infty}\frac{1}{n}z^{-n}$$
$$=\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{1}{z}\right]^n$$

$$= \left[\frac{1}{z}\right] + \frac{\left[\frac{1}{z}\right]^2}{2} + \frac{\left[\frac{1}{z}\right]^3}{3} + \ldots$$

$$= -\log\left[1 - \frac{1}{z}\right]$$

$$= -\log\left[\frac{z-1}{z}\right]$$

$$= \log\left[\frac{z-1}{z}\right]^{-1}$$

$$= \log\left[\frac{z}{z-1}\right], \ |z| > 1$$

44. Find the inverse Z-Transform of $\left[\dfrac{z}{(z+1)^2}\right]$ **[A.U. N/D 2013]**

Solution:

$$\text{Let} \quad F(Z) = \left[\frac{z}{(z+1)^2}\right]$$

$$\frac{F(Z)}{z} = \left[\frac{1}{(z+1)^2}\right]$$

$$\left[\frac{1}{(z+1)^2}\right] = \frac{A}{z+1} + \frac{B}{(z+1)^2}$$

$$1 = A(z+1) + B$$

$$put \ \ z = -1, \ then \ \ B = 1$$
$$put \ \ z = 0, \ then \ \ A = 0$$

$$\frac{F(Z)}{z} = \left[\frac{1}{(z+1)^2}\right]$$

$$F(Z) = \left[\frac{z}{(z+1)^2}\right]$$

$$Z^{-1}\left[\frac{z}{(z+1)^2}\right] = -Z^{-1}\left[\frac{-z}{(z-(-1))^2}\right]$$

$$= -n(-1)^n \qquad \because Z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n$$

45. Find the inverse Z-Transform of $\left[\dfrac{z}{(z+1)(z+2)}\right]$ **[A.U. M/J 2013]**

Solution:

Consider $\left[\dfrac{z}{(z+1)(z+2)}\right] = \dfrac{A}{z+1} + \dfrac{B}{z+2}$

$$z = A(z+2) + B(z+1)$$
$$put\ \ z = -2,\ then\ B = 2$$
$$put\ \ z = -1,\ then\ A = -1$$

$$Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right] = Z^{-1}\left[\frac{-1}{(z+1)}\right] + Z^{-1}\left[\frac{2}{(z+2)}\right]$$

$$= -(-1)^{n-1} + 2(-2)^{n-1}$$

$$= (-1)^{n} + 2(-2)^{n-1}$$

46. Find the Z-Transform of $\dfrac{1}{n+1}$ [A.U. N/D 2013, 2015, A/M 2018]

Solution:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1}\left[\frac{1}{z}\right]^{n}$$

$$= 1 + \frac{\left[\frac{1}{z}\right]}{2} + \frac{\left[\frac{1}{z}\right]^{2}}{3} + ...$$

$$= z\left[\frac{1}{z}\right]\left[1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^{2} + ...\right]$$

$$= z\left[\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^{2} + ...\right]$$

$$= z\left[-\log\left(1 - \frac{1}{z}\right)\right]$$

$$= z\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$= z\log\left[\frac{z-1}{z}\right]^{-1}$$

$$= z\log\left[\frac{z}{z-1}\right],\ \ |z| > 1$$

47. Find the Z-Transform of $\frac{1}{n!}$ **[A.U. N/D 2011, M/J 2016]**

Solution:

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{1}{z}\right]^{n}$$

$$= 1 + \frac{1}{1!}\left[\frac{1}{z}\right] + \frac{1}{2!}\left[\frac{1}{z}\right]^{2} + \dots\dots\dots$$

$$= e^{\frac{1}{z}}$$

48. Find $Z\left(e^{-t}t^{2}\right)$ **[A.U. N/D 2016]**

Solution:

$$Z\left(e^{-t}t^{2}\right) = \left[Z\left(t^{2}\right)\right]_{z \to ze^{T}}$$

$$= \left[\frac{T^{2}z(z+1)}{(z-1)^{3}}\right]_{z \to ze^{T}}$$

$$= \frac{T^{2}ze^{T}\left(ze^{T}+1\right)}{\left(ze^{T}-1\right)^{3}}$$

49. Form a difference equation by eliminating the arbitrary constant 'a' from $y_n = a.2^n$. **[AU N/D 2017]**

Solution:

$$y_n = a.2^n.$$

$$y_{n+1} = a.2^{n+1}.$$

$$= 2a.2^{n}$$

$$= 2y_n$$

$$y_{n+1} - 2y_n = 0$$

PART-B

PROBLEMS BASED ON CONVOLUTION THEOREM

1. Using Convolution Theorem, Find the $Z^{-1}\left[\dfrac{z^2}{(z-4)(z-3)}\right]$

Solution: **[AU N/D 2010, 2015]**

$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right]=Z^{-1}\left[\frac{z}{z-4}*\frac{z}{z-3}\right]$$
$$=Z^{-1}\left[\frac{z}{z-4}\right]*Z^{-1}\left[\frac{z}{z-3}\right]$$
$$=(4)^n*(3)^n$$
$$=\sum_{r=0}^{n}(3)^r.(4)^{n-r}$$
$$=(4)^n\left[\sum_{r=0}^{n}(3)^r.(4)^{-r}\right]$$
$$=(4)^n\left[\sum_{r=0}^{n}\left(\frac{3}{4}\right)^r\right]$$
$$=(4)^n\left[1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\dots+\left(\frac{3}{4}\right)^n\right]$$
$$=(4)^n\left[\frac{\left(\frac{3}{4}\right)^{n+1}-1}{\frac{3}{4}-1}\right]\qquad\left[\begin{array}{l}formula:\\ 1+a+a^2+....+a^{n-1}=\dfrac{a^n-1}{a-1}\end{array}\right]$$
$$=(4)^n\left[\frac{\dfrac{3^{n+1}-4^{n+1}}{4^{n+1}}}{-\dfrac{1}{4}}\right]$$
$$=-\left[3^{n+1}-4^{n+1}\right]$$

Result: $=\left[4^{n+1}-3^{n+1}\right]$

2. Using convolution theorem, find the value of $Z^{-1}\left[\dfrac{z^3}{(z-2)^2(z-3)}\right]$

Solution: **[AU N/D 2010]**

$$Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right]=Z^{-1}\left[\frac{z^2}{(z-2)^2}.\frac{z}{(z-3)}\right]$$
$$=Z^{-1}\left[\frac{z^2}{(z-2)^2}\right]*Z^{-1}\left[\frac{z}{z-3}\right]$$
$$=(n+1)2^n*3^n$$

$$= \sum_{r=0}^{n} (r+1) 2^r 3^{n-r}$$

$$= 3^n \sum_{r=0}^{n} (r+1) \left(\frac{2}{3}\right)^r$$

$$= 3^n \left[1 + 2\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)^3 + ... + (n+1)\left(\frac{2}{3}\right)^n \right]$$

$$\text{Let } S = 1 + 2(x) + 3(x)^2 + 4(x)^3 + ... + (n+1)(x)^n, \quad \text{where } x = \frac{2}{3}$$

$$S = 1 + 2x + 3x^2 + 4x^3 + ... + nx^{n-1} + (n+1)x^n$$

$$xS = x + 2x^2 + 3x^3 + 4x^4 + ... + nx^n + (n+1)x^{n+1}$$

$$(1-x)S = \left(1 + x + x^2 + x^3 + 4x^4 + ... + x^n\right) - (n+1)x^{n+1}$$

$$= \frac{1 - x^{n+1}}{1 - x} - (n+1)x^{n+1}$$

$$S = \frac{1 - x^{n+1}}{(1-x)^2} - \frac{(n+1)x^{n+1}}{1-x}$$

$$\therefore\ S = 9\left[1 - \left(\frac{2}{3}\right)^{n+1}\right] - 3(n+1)\left(\frac{2}{3}\right)^{n+1} \qquad \text{Since } x = \frac{2}{3},\ 1 - x = \frac{1}{3}$$

$$= 9 - \left(\frac{2}{3}\right)^{n+1} [9 + 3n + 3]$$

$$= 9 - \left(\frac{2}{3}\right)^{n+1} [12 + 3n]$$

$$= 9 - 3[n+4]\left(\frac{2}{3}\right)^{n+1}$$

$$\therefore \quad Z^{-1}\left[\frac{z^3}{(z-2)^2 (z-3)}\right] = 3^n \left\{ 9 - 3[n+4]\left(\frac{2}{3}\right)^{n+1} \right\}$$

$$= 9.3^n - 3^{n+1}(n+4)\left(\frac{2}{3}\right)^{n+1}$$

Result: $= 3^{n+2} - (n+4)2^{n+1}$

3. Using convolution theorem finds the inverse Z- transform of $\left[\frac{14z^2}{(7z-1)(2z-1)}\right]$

Solution:

$$Z^{-1}\left[\frac{14z^2}{(7z-1)(2z-1)}\right] = Z^{-1}\left[\frac{z^2}{\left(z - \frac{1}{7}\right)\left(z - \frac{1}{2}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{7}\right)}\cdot\frac{z}{\left(z-\frac{1}{2}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{7}\right)}\right] * Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right]$$

$$= \left(\frac{1}{7}\right)^n * \left(\frac{1}{2}\right)^n$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{7}\right)^n$$

$$= \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{n-r}\left(\frac{1}{7}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{-r}\left(\frac{1}{7}\right)^r$$

$$= \left(\frac{1}{2}\right)^n\left[1+\frac{2}{7}+\left(\frac{2}{7}\right)^2+\left(\frac{2}{7}\right)^3+\ldots.+\left(\frac{2}{7}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{2}{7}\right)^{n+1}}{1-\frac{2}{7}}\right] \qquad \left[\begin{array}{l}Formula:\\ 1+a+a^2+a^3+\ldots.+a^{n-1}=\dfrac{1-a^n}{1-a}\end{array}\right]$$

$$= \left(\frac{1}{2}\right)^n\frac{7}{5}\left[1-\left(\frac{2}{7}\right)^{n+1}\right]$$

$$= \frac{7}{5}\left(\frac{1}{2}\right)^n - \frac{7}{5}\times\frac{2}{7}\left(\frac{2}{7}\right)^n\left(\frac{1}{2}\right)^n$$

Result: $= \frac{7}{5}\left(\frac{1}{2}\right)^n - \frac{2}{5}\left(\frac{1}{7}\right)^n$

4. Using Convolution Theorem Find the value of $Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\right]$

Solution: **[AU A/M 2012, N/D 2017]**

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\right] = Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\cdot\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$

$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right] * Z^{-1}\left[\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{n-r}\left(\frac{1}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{-r}\left(\frac{1}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{-r}\left(\frac{1}{2}\right)^{2r}$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{-r+2r}$$

$$= \left(\frac{1}{2}\right)^n \left[1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+.....+\left(\frac{1}{2}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}\right] \qquad \left[\begin{array}{l} Formula: \\ 1+a+a^2+a^3+....+a^{n-1} = \dfrac{1-a^n}{1-a} \end{array}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1}\left[1-\left(\frac{1}{2}\right)^{n+1}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{n-1+n+1}$$

Result: $= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n}$

5. Using Convolution theorem, Find $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$ **[AU N/D 2007, M/J 2016]**

(OR)

Find $Z^{-1}\left[(1-az^{-1})^{-2}\right]$ **[AU A/M 2015]**

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a}.\frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= (a)^n * (a)^n$$

$$= \sum_{r=0}^{n} (a)^r . (a)^{n-r}$$
$$= a^n + a.a^{n-1} + ... + a^n$$
$$= a^n + a^n + ... + a^n$$

Result: $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n$

6. Using convolution theorem find the inverse Z- transform of $\left[\frac{z^2}{(z-1)(z-3)}\right]$

Solution: **[AU N/D 2006, 2013]**

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1} . \frac{z}{z-3}\right]$$
$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$
$$= 1^n * 3^n$$
$$= \sum_{r=0}^{n} (1)^r (3)^{n-r} \quad \left[\because z(1^n) = \frac{z}{z-1}\right] \quad \left[\because z(3^n) = \frac{z}{z-3}\right]$$
$$= 3^n + 3^{n-1} + 3^{n-2} + ... + 3^1 + 1$$
$$= 1 + 3 + ... + 3^n$$
$$= \frac{3^{n+1} - 1}{3 - 1}$$

Result: $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{3^{n+1} - 1}{2}$

7. Find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$**, Using Convolution Theorem. [AU A/M 2010, 2017, 2018]**

Solution:

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}\right]$$
$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)} . \frac{z}{\left(z+\frac{1}{4}\right)}\right]$$
$$= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)} .\right] * Z^{-1}\left[\frac{z}{\left(z+\frac{1}{4}\right)}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{-1}{4}\right)^n$$

$$= \sum_{r=0}^{n} \left(\frac{1}{2}\right)^{n-r} \left(\frac{-1}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{1}{2}\right)^{-r} \left(\frac{-1}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} 2^r \left(\frac{-1}{4}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n} \left(\frac{-1}{2}\right)^r$$

$$= \left(\frac{1}{2}\right)^n \left\{\frac{1-\left(\frac{-1}{2}\right)^{n+1}}{1-\left(\frac{-1}{2}\right)}\right\}$$

$$= \left(\frac{1}{2}\right)^n \left\{\frac{1-\left(\frac{-1}{2}\right)^{n+1}}{\left(\frac{3}{2}\right)}\right\}$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n \left\{1-\left(\frac{-1}{2}\right)^n\left(\frac{-1}{2}\right)\right\}$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n \left\{1+\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)^n\right\}$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n\left(\frac{-1}{2}\right)^n$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\left(\frac{1}{2}\right)\times\left(\frac{-1}{2}\right)\right)^n$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{-1}{4}\right)^n$$

Result: $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{-1}{4}\right)^n$

8. Using Convolution Theorem, Find the $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Solution: **[AU A/M 2013, 2014 N/D 2016]**

$$z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = z^{-1}\left[\frac{z}{z-a} * \frac{z}{z-b}\right]$$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right]$$

$$= (a)^n * (b)^n$$

$$= \sum_{r=0}^{n} (a)^r .(b)^{n-r}$$

$$= (b)^n \left[\sum_{r=0}^{n} \left(\frac{a}{b}\right)^r \right]$$

$$= (b)^n \left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \ldots + \left(\frac{a}{b}\right)^n \right]$$

$$= (b)^n \left[\frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}} \right] \quad \left[\begin{array}{l} formula: \\ a + ar + ar^2 + \ldots. + ar^n = \dfrac{a(1-r^{n+1})}{1-r} \end{array} \right]$$

$$= \frac{b^n \left[\dfrac{(b)^{n+1} - a^{n+1}}{b^{n+1}} \right]}{\dfrac{b-a}{b}}$$

Result: $$= \frac{(b)^{n+1} - a^{n+1}}{b-a}$$

PROBLEMS BASED ON DIFFERENCE EQUATION

9. Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ **with** $y_0 = 0$ **and** $y_1 = 1$ **using Z-transform**

Solution: **[AU N/D 2010, A/M 2005,2006]**

Taking Z-transform, we get

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z(2^n)$$

$$z^2 \bar{y} - z^2 y_0 - zy_1 + 4[z\bar{y} - zy_0] + 3\bar{y} = \frac{z}{z-2}$$

Given $y_0 = 0$, $y_1 = 1$

$$z^2 \bar{y} - z + 4z\bar{y} + 3\bar{y} = \frac{z}{z-2}$$

$$\bar{y}(z^2 + 4z + 3) = \frac{z}{z-2} + z$$

$$\bar{y} = \frac{z}{(z-2)(z+1)(z+3)} + \frac{z}{(z+1)(z+3)}$$

Now, $$\frac{z}{(z-2)(z+1)(z+3)} = \frac{A}{(z-2)} + \frac{B}{(z+1)} + \frac{C}{(z+3)}$$

Put $z = -1$	put $z = -3$	put $z = 2$
$-1 = -6B$	$-3 = 10c$	$2 = 15A$
$B = \frac{1}{6}$	$C = \frac{-3}{10}$	$A = \frac{2}{15}$

$$\frac{z}{(z-2)(z+1)(z+3)} = \frac{2}{15}\frac{1}{z-2} + \frac{1}{6}\frac{1}{z+1} - \frac{3}{10}\frac{1}{z+3}$$

$$\frac{z}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$z = A(z+3) + B(z+1)$$

Put $z=-3$	put $z=-1$
$-3=-2B$	$-1=2A$
$B=\frac{3}{2}$	$A=\frac{-1}{2}$

$$\therefore \frac{z}{(z+1)(z+3)} = \frac{-1}{2}\frac{z}{z+1} + \frac{3}{2}\frac{1}{z+3}$$

$$\overline{Y} = \frac{2}{15}\frac{1}{z-2} + \frac{1}{6}\frac{1}{z+1} - \frac{3}{10}\frac{1}{z+3} - \frac{1}{2}\frac{1}{z+1} + \frac{3}{2}\frac{1}{z+3}$$

$$Z[Y_n] = \frac{2}{15}\frac{1}{z-2} + \frac{1}{6}\frac{1}{z+1} - \frac{3}{10}\frac{1}{z+3} - \frac{1}{2}\frac{1}{z+1} + \frac{3}{2}\frac{1}{z+3}$$

$$y_n = \frac{2}{15}Z^{-1}\left[\frac{1}{Z-2}\right] + \frac{1}{6}Z^{-1}\left[\frac{1}{Z+1}\right] - \frac{3}{10}Z^{-1}\left[\frac{1}{Z+3}\right] - \frac{1}{2}Z^{-1}\left[\frac{1}{Z+1}\right] + \frac{3}{2}Z^{-1}\left[\frac{1}{Z+3}\right]$$

$$y_n = \frac{2}{15}2^{n-1} + \frac{1}{6}(-1)^{n-1} - \frac{3}{10}(-3)^{n-1} - \frac{1}{2}(-1)^{n-1} + \frac{3}{2}(-3)^{n-1}$$

$$= \frac{1}{15}2^n + \left(\frac{1}{6} - \frac{1}{2}\right)(-1)^{n-1} + (-3)^{n-1}\left[\frac{3}{2} - \frac{3}{10}\right]$$

$$= \frac{2^n}{15} + \left(\frac{-2}{6}\right)(-1)^{n-1} + (-3)^{n-1}\left(\frac{12}{10}\right)$$

Result: $= \frac{2^n}{15} + \left(\frac{1}{3}\right)(-1)^n - \left(\frac{2}{5}\right)(-3)^n$

10. Solve the Equation $y_{n+2} - 3y_{n+1} + 2y_n = 2^n$ **given that** $y_0 = y_1 = 0$

Solution: **[AU A/M 2008, 2009, 2011]**

$$y(n+2) - 3y(n+1) + 2y(n) = \left(2^n\right) \qquad \ldots (1)$$

Taking Z-Transform on both sides, we get,

$$Z[y(n+2)] - 3Z[y(n+1)] + 2Z[y(n)] = Z\left(2^n\right)$$

$$z^2\overline{y} - z^2 y(0) - zy(1) - 3z\left[\overline{y} - y(0)\right] + 2\overline{y} = \frac{z}{z-2} \qquad \ldots. (2) \qquad where \quad \overline{y} = Z[y(n)]$$

Applying y (0) =0 and y (1) =0 in (2)

$$z^2\left[\overline{y}\right] - 3z\overline{y} + 2\overline{y} = \frac{z}{z-2}$$

$$\overline{y}\left[z^2 - 3z + 2\right] = \frac{z}{z-2}$$

$$\overline{y} = \frac{z}{\left(z^2 - 3z + 2\right)(z-2)}$$

$$= \frac{z}{(z-2)(z-1)(z-2)}$$

$$= \frac{z}{(z-1)(z-2)^2}$$

$$Z[y(n)] = \frac{z}{(z-1)(z-2)^2}$$

$$\frac{y(n)}{z} = Z^{-1}\left[\frac{1}{(z-1)(z-2)^2}\right] \qquad \text{…. (3)}$$

$$\frac{1}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \qquad \text{…. (4)}$$

$$1 = A(z-2)^2 + B(z-2)(z-1) + c(z-1)$$

Put $z=1$	put $z=2$	Equating Coefficient of z^2
$A=1$	$C=1$	$A+B=0$
		$B=-1$

$\therefore$ Substituting A=1; B= -1; C=1 in (3), we get

$$\frac{1}{(z-1)(z-2)^2} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{(z-2)^2}$$

$$\therefore Z^{-1}\left[\frac{z}{(z-1)(z-2)^2}\right] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{(z-2)^2}\right]$$

$$y(n) = (1)^{n-1} - (2)^n + \frac{1}{2}Z^{-1}\left[\frac{2z}{(z-2)^2}\right] \quad , \quad n \geq 1$$

$$= 1 - 2^n + \frac{1}{2}n2^n$$

Result: $y(n) = 1 - 2^n + n2^{n-1}; \quad n \geq 0$

11. Using Z-transform solve $U(n+2) - 5U(n+1) + 6U(n) = 4^n$ **given that** $U(0) = 0, U(1) = 1$ **[AU A/M 2005 N/D 2009]**

Solution:

$$U(n+2) - 5U(n+1) + 6U(n) = 4^n$$

Taking Z-transform on both sides, we get

$$Z[U(n+2)] - 5Z[U(n+1)] + 6Z[U(n)] = Z[4^n]$$

$$z^2\overline{U} - z^2U(0) - ZU(1) - 5Z[\overline{U} - U(0)] + 6\overline{U} = \frac{z}{z-4} \quad , \quad \text{Where } \overline{U} = Z[U(n)]$$

Applying U (0) =0 and U (1) =1

$$z^2\overline{U} - Z - 5Z\overline{U} + 6\overline{U} = \frac{z}{z-4}$$

$$\overline{U}[z^2 - 5Z + 6] = \frac{z}{z-4} + z$$

$$\overline{U}[(z-2)(z-3)] = \frac{z}{z-4} + z$$

$$\overline{U} = \frac{z^2 - 3z}{(z-2)(z-3)(z-4)}$$

$$\frac{\overline{U}}{z} = \frac{z-3}{(z-2)(z-3)(z-4)} \quad \text{…. (3)}$$

Now $\frac{z-3}{(z-2)(z-3)(z-4)} = \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-4)}$

$$z - 3 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

$Put\ z = 3$ $B = 0$	$Put\ z = 4$ $1 = 2C$ $C = \frac{1}{2}$	$Put\ z = 2$ $-1 = A(-1)(-2)$ $A = -\frac{1}{2}$

$$\frac{z-3}{(z-2)(z-3)(z-4)} = \frac{-1}{2}\frac{1}{(z-2)} + \frac{1}{2}\frac{1}{(z-4)} \quad \text{…. (4)}$$

$$\overline{U} = \frac{-1}{2}\frac{z}{(z-2)} + \frac{1}{2}\frac{z}{(z-4)}$$

$$Z[U(n)] = \frac{-1}{2}\frac{z}{z-2} + \frac{1}{2}\frac{z}{z-4}$$

$$U(n) = \frac{-1}{2}Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{2}Z^{-1}\left[\frac{z}{z-4}\right]$$

Result: $= \frac{-1}{2}(2)^n + \frac{1}{2}(4)^n \quad , n \geq 1$

12. Solve by using Z –transform $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$, given that $y_0 = 0$; $y_1 = 1$.

Solution: **[AU A/M 2009, 2010 N/D 2015]**

Taking Z-transform on both sides, we get

$$Z[y(n+2)] + 4Z[y(n+1)] + 3Z[y(n)] = Z[3^n]$$

$$z^2\overline{y} - z^2y(0) - zy(1) + 4z[\overline{y} - y(0)] + 3\overline{y} = \frac{z}{z-3} \quad \text{…. (1)} \qquad \text{Where } \overline{y} = Z[y(n)]$$

Given $y_0 = 0$; $y_1 = 1$ …. (2)

Applying (2) in (1), we get

$$\overline{y}[z^2 + 4z + 3] = \frac{z}{z-3} + z$$

$$= \frac{z + z(z-3)}{z-3}$$

$$= \frac{z^2 - 2z}{z-3}$$

$$\overline{y} = \frac{z(z-2)}{(z+1)(z+3)(z-3)}$$

$$\frac{\overline{y}}{z} = \frac{(z-2)}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3}$$

$$z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3)$$

Put z =-1	Put z = -3	Put z = 3
-3= -8A	-5=12B	1=24C
$A = \frac{3}{8}$	$B = \frac{-5}{12}$	$C = \frac{1}{24}$

$$\frac{\bar{y}}{z} = \frac{\frac{3}{8}}{z+1} + \frac{\frac{-5}{12}}{z+3} + \frac{\frac{1}{24}}{z-3}$$

$$y(n) = \frac{3}{8}Z^{-1}\left[\frac{z}{(z+1)}\right] - \frac{5}{12}Z^{-1}\left[\frac{z}{(z+3)}\right] + \frac{1}{24}Z^{-1}\left[\frac{z}{(z-3)}\right]$$

Result: $= \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}(3)^n$

13. Solve the difference equation $y(k+2) - 4y(k+1) + 4y(k) = 0$ **where** $\mathbf{y(0)=1,\ y(1)=0}$.

(or) Solve $y_{n+2} - 4y_{n+1} + 4y_n = 0$ **,** $\mathbf{y(0)=1,\ y(1)=0}$**, using Z-transform**

[AU N/D 2005, 2009, A/M 2009, 2018]

Solution:

Given: $y(k+2) - 4y(k+1) + 4y(k) = 0$

Taking Z transform on both sides, we get

$$Z[y(k+2)] - 4Z[y(k+1)] + 4Z[y(k)] = 0$$

$$z^2\bar{y} - z^2 y(0) - zy(1) - 4z[\bar{y} - y(0)] + 4\bar{y} = 0, \quad \text{Where } \bar{y} = Z[y(k)]$$

Given y (0) =1, y (1) =0

$$z^2[\bar{y} - 1] - 4z[\bar{y} - 1] + 4\bar{y} = 0$$

$$\bar{y}[z^2 - 4z + 4] - z^2 + 4z = 0$$

$$\bar{y} = \frac{z^2 - 4z}{(z^2 - 4z + 4)}$$

$$\frac{\bar{y}}{z} = \frac{z-4}{(z-2)^2}$$

$$\frac{z-4}{(z-2)^2} = \frac{A}{(z-2)} + \frac{B}{(z-2)^2} \qquad \text{…. (3)}$$

$$z-4 = A(z-2) + B$$

Put $z = 2$	Put $z = 0$
$B = -2$	$-2A + B = -4$
	$A = 1$

Sub. A=1 and B=-2 in (1), we get

$$\frac{\bar{y}}{z} = \frac{1}{z-2} - \frac{2}{(z-2)^2}$$

$$\bar{y} = \frac{z}{z-2} - \frac{2z}{(z-2)^2}$$

$$Z[y(n)] = \frac{z}{z-2} - \frac{2z}{(z-2)^2}$$

$$y(n) = Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= 2^n - n2^n \qquad \left[\because z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n\right]$$

Result: $= 2^n(1-n)$

14. Solve the difference equation $y(n) + 3y(n-1) - 4y(n-2) = 0$ **,** $n \geq 2$**, given that y(0)=3,y(1)=-2.** **[AU A/M 2006]**

Solution:

Given that

$$y(n) + 3y(n-1) - 4y(n-2) = 0$$

Changing n into n+2, the given equation becomes,

$$y(n+2) + 3y(n+1) - 4y(n) = 0 \quad \ldots.(1) \quad \text{, where } \bar{y} = Z[y(n)]$$

$$Z(y_{n+2}) + 3Z(y_{n+1}) - 4Z(y_n) = 0$$

$$[z^2\bar{y} - z^2 y(0) - zy(1)] + 3[z\bar{y} - zy(0)] - 4\bar{y} = 0$$

Applying y (0) =3 and y (1) =-2 in (2), we get

$$(z^2 + 3z - 4)\bar{y} = 3z^2 + 7z$$

$$\bar{y} = \frac{3z^2 + 7z}{(z+4)(z-1)}$$

$$\frac{\bar{y}}{z} = \frac{3z+7}{(z+4)(z-1)}$$

Now $\dfrac{3z+7}{(z+4)(z-1)} = \dfrac{A}{z+4} + \dfrac{B}{z-1}$

$$3Z + 7 = A(z-1) + B(z+4)$$

Put z =1 Put z = -4

5B = 10 -5A = -5

B = 2 A =1

$$\frac{3z+7}{(z+4)(z-1)} = \frac{1}{z+4} + \frac{2}{z-1} \quad \ldots.(3)$$

$$\frac{\bar{y}}{z} = \frac{1}{z+4} + \frac{2}{z-1}$$

$$\bar{y} = \frac{z}{z+4} + \frac{2z}{z-1}$$

$$Z[y(n)] = \frac{z}{z+4} + \frac{2z}{z-1}$$

$$[y(n)] = Z^{-1}\left[\frac{z}{z+4}\right] + Z^{-1}\left[\frac{2z}{z-1}\right]$$

$$= Z^{-1}\left[\frac{z}{z+4}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right]$$

Result: $y(n) = (-4)^n + 2(1)^n \quad , n \geq 0$

15. Solve using Z-transform, $y_{n+2} - 3y_{n+1} - 10y_n = 0$ given $y_0 = 1$ and $y_1 = 0$

[Dec.2009,2008, Apr.2004, Nov 2002, May/June (2013,2014)]

Solution:

Given that

$$y_{n+2} - 3y_{n+1} - 10y_n = 0$$

Taking Z-transform, we get

$$Z[y(n+2)] - 3Z[y(n+1)] - 10Z[y(n)] = 0$$

$$z^2\bar{y} - z^2 y(0) - zy(1) - 3[z\bar{y} - zy(0)] - 10\bar{y} = 0 \quad \text{.... (1)} \quad \text{Where } \bar{y} = Z[y(n)]$$

Given $y(0) = 1,\ y(1) = 0$ ....(2)

Substituting (2) in(1), we get

$$z^2[\bar{y} - 1] - 3z[\bar{y} - 1] - 10\bar{y} = 0$$

$$\bar{y}[z^2 - 3z - 10] - z^2 + 3z = 0$$

$$\bar{y}[z^2 - 3z - 10] = z^2 - 3z$$

$$\bar{y} = \frac{z(z-3)}{z^2 - 3z - 10}$$

$$= \frac{z(z-3)}{(z-5)(z+2)}$$

$$\frac{\bar{y}}{z} = \frac{(z-3)}{(z-5)(z+2)} \quad \text{....(3)}$$

$$\frac{(z-3)}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2}$$

$$(z-3) = A(z+2) + B(z-5)$$

Put $z = -2$	Put $z = 5$
$-5 = -7B$	$2 = 7A$
$B = \frac{5}{7}$	$A = \frac{2}{7}$

$$\frac{(z-3)}{(z-5)(z+2)} = \frac{\frac{2}{7}}{z-5} + \frac{\frac{5}{7}}{z+2}$$

$$\frac{\bar{y}}{z} = \frac{\frac{2}{7}}{z-5} + \frac{\frac{5}{7}}{z+2} \quad \text{[Using (3)]}$$

$$\bar{y} = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

$$Z[y(n)] = \frac{2}{7}\frac{z}{z-5} + \frac{5}{7}\frac{z}{z+2}$$

Result: $$y(n) = \frac{2}{7}Z^{-1}\left[\frac{z}{z-5}\right] + \frac{5}{7}Z^{-1}\left[\frac{z}{z+2}\right]$$

$$= \frac{2}{7}(5)^n + \frac{5}{7}(-2)^n \quad , n \geq 0$$

16. Solve the difference equation $y(k+2)+y(k)=1$, $y(0)=y(1)=0$, using Z-transform **[AU A/M 2012]**

Solution:

Given $y(k+2)+y(k)=1$, $y(0)=y(1)=0$

i.e., $y(n+2)+y(n)=1$

Taking Z-transform on both sides , we get

$$Z[y(n+2)]+Z[y(n)]=Z[1]$$

$$\left[z^2 y(z)-z^2 y(0)-zy(1)\right]+y(z)=\frac{z}{z-1}$$

$$z^2 y(z)+y(z)=\frac{z}{z-1} \qquad [\because y(0)=y(1)=0]$$

$$\left(z^2+1\right)y(z)=\frac{z}{z-1}$$

$$y(z)=\frac{z}{(z-1)\left(z^2+1\right)}$$

$$y(z)z^{n-1}=\frac{z}{(z-1)\left(z^2+1\right)}z^{n-1}$$

$$y(z)z^{n-1}=\frac{z^n}{(z-1)(z+i)(z-i)}$$

z = 1 is a simple pole

z = i is a simple pole

z = -i is a simple pole

$$\operatorname*{Res}_{z=1} y(z)z^{n-1}=\lim_{z\to 1}(z-1)\frac{z^n}{(z-1)(z+i)(z-i)}$$

$$=\lim_{z\to 1}\frac{z^n}{(z+i)(z-i)}$$

$$=\frac{1}{(1+i)(1-i)}=\frac{1}{2}$$

$$\operatorname*{Res}_{z=i} y(z)z^{n-1}=\lim_{z\to i}(z-i)\frac{z^n}{(z-1)(z+i)(z-i)}$$

$$=\lim_{z\to i}\frac{z^n}{(z-1)(z+i)}$$

$$=\frac{(i)^n}{(i-1)(i+i)}$$

$$=\frac{(i)^n}{2i(i-1)}$$

$$\operatorname*{Res}_{z=-i} y(z)z^{n-1}=\lim_{z\to -i}(z+i)\frac{z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z \to -i} \frac{z^n}{(z-1)(z-i)}$$

$$= \frac{(-i)^n}{(-i-1)(-i-i)}$$

$$= \frac{(-i)^n}{2i(1+i)}$$

Result: $\therefore\ y(n) =$ sum of the residues

$$= \frac{1}{2} + \frac{(i)^n}{2i(i-1)} + \frac{(-i)^n}{2i(1+i)}$$

$$= \frac{1}{2} + \frac{i^n}{2i}\left[\frac{1}{i-1} + \frac{(-1)^n}{1+i}\right]$$

$$= \frac{1}{2} + \frac{i^n}{2i}\left[\frac{(1+i)+(i-1)(-1)^n}{-2}\right]$$

$$= \frac{1}{2} - \frac{i^n}{4i}\left[(i+1)+(-1)^n(i-1)\right]$$

$$= \frac{1}{2} - \frac{i^{n-1}}{4}\left[(i+1)+(-1)^n(i-1)\right].$$

17. Solve the difference equation $y_{n+2} + y_n = 2$, $y_0 = y_1 = 0$, using Z-transform. **[AU M/J 2016]**

Solution:

Given $y(n+2) + y(n) = 2$

Taking Z-transform on both sides , we get

$$Z[y(n+2)] + Z[y(n)] = Z[2]$$

$$\left[z^2 y(z) - z^2 y(0) - zy(1)\right] + y(z) = \frac{2z}{z-1}$$

$$z^2 y(z) + y(z) = \frac{2z}{z-1} \qquad [\because y(0) = y(1) = 0]$$

$$(z^2+1)y(z) = \frac{2z}{z-1}$$

$$y(z) = \frac{2z}{(z-1)(z^2+1)}$$

$$\frac{y(z)}{z} = \frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$2 = A(z^2+1) + (Bz+C)(z-1)$$

put $z = 0$, *we get*

$2 = A - C$

$C = A - 2$

$C = 1 - 2$

$C = -1$

put $z = 1$, *we get*

$2 = 2A$

$A = 1$

equating the coefficients of z^2 *on both sides, we get*

$0 = A + B$

$B = -1$

$$\therefore \quad (1) \Rightarrow \quad \frac{y(z)}{z} = \frac{1}{z-1} + \frac{-z-1}{z^2+1}$$

$$y(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$$

$$Z[y(n)] = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$$

$$y(n) = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right] - Z^{-1}\left[\frac{z}{z^2+1}\right]$$

Result: $y(n) = (1)^n - \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}$

18. Solve $y_{n+2} + y_n = n2^n$, using Z-transform **[AU A/M 2012]**

Solution:

Given $y_{n+2} + y_n = n.2^n$

$$Z[y_{n+2}] + Z[y_n] = Z[n.2^n]$$

$$[z^2 y(z) - z^2 y(0) - zy(1)] + y(z) = \frac{2z}{(z-2)^2} \qquad \left[\because Z[na^n] = \frac{az}{(z-a)^2}\right]$$

$$(z^2+1)y(z) - z^2 A - zB = \frac{2z}{(z-2)^2}$$

When $A = y(0)$, $B = y(1)$

$$(z^2+1)y(z) = \frac{2z}{(z-2)^2} + z^2 A + zB$$

$$y(z) = \frac{2z}{(z-2)^2(z^2+1)} + \frac{Az^2}{(z^2+1)} + \frac{Bz}{(z^2+1)}$$

$$y(z)z^{n-1} = \frac{2z^n}{(z-2)^2(z^2+1)} + \frac{Az^{n+1} + Bz^n}{(z^2+1)}$$

$$= I_1 + I_2$$

Step 1:

$$I_1 = \frac{2z^n}{(z-2)^2(z^2+1)}$$

$$= \frac{2z^n}{(z-2)^2(z+i)(z-i)}$$

$z = 2$ is a pole of order 2

$z = i$ is a simple pole

$z = -i$ is a simple pole

$$\operatorname*{Res}_{z=2} I_1 = \lim_{z\to 2}\frac{d}{dz}(z-2)^2\frac{2z^n}{(z-2)^2(z^2+1)}$$

$$= \lim_{z\to 2}\frac{d}{dz}\frac{2z^n}{z^2+1}$$

$$= \lim_{z\to 2}\frac{(z^2+1)(2n\,z^{n-1})-2z^n(2z)}{(z^2+1)^2}$$

$$= \frac{(5)\,2n\,2^{n-1}-2\,2^n(4)}{25}$$

$$= \frac{5n2^n-2^{n+1}\,2^2}{25}$$

$$= \frac{2^n\left[5n-2^3\right]}{25}$$

$z=i$ is a pole of order 1

$$\operatorname*{Res}_{z=i} I_1 = \lim_{z\to i}(z-i)\frac{2z^n}{(z-2)^2(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{2z^n}{(z-2)^2(z+i)}$$

$$= \frac{2i^n}{(i-2)^2(2i)}$$

$$= \frac{i^{n-1}}{(i-2)^2}$$

$z=-i$ is a simple pole

$$\operatorname*{Res}_{z=-i} I_1 = \lim_{z\to -i}(z+i)\frac{2z^n}{(z-2)^2(z+i)(z-i)}$$

$$= \lim_{z\to -i}\frac{2z^n}{(z-2)^2(z-i)}$$

$$= \frac{2(-i)^n}{(-i-2)^2(-2i)} \qquad = \frac{(-i)^{n-1}}{(i+2)^2}$$

<u>Step 2:</u>

$$I_2 = \frac{Az^{n+1}+Bz^n}{z^2+1}$$

$z=i$ is a simple pole

$z=-i$ is a simple pole

$$\operatorname*{Res}_{z=i} I_2 = \lim_{z\to i}(z-i)\frac{Az^{n+1}+Bz^n}{(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{Az^{n+1}+Bz^n}{(z+i)}$$

$$= \frac{Ai^{n+1} + Bi^{n}}{2i}$$
$$= \frac{i^{n}\left(Ai + B\right)}{2i}$$
$$= \frac{1}{2} i^{n-1}\left(Ai + B\right)$$

$$\operatorname*{Re}_{z=-i} s\, I_2 = \lim_{z\to -i} (z+i)\frac{Az^{n+1} + Bz^{n}}{(z+i)(z-i)}$$
$$= \lim_{z\to -i} \frac{Az^{n+1} + Bz^{n}}{(z-i)}$$
$$= \frac{A(-i)^{n+1} + B(-i)^{n}}{-2i}$$
$$= \frac{1}{2} A(-i)^{n} + \frac{B}{2}(-i)^{n-1}$$

Result: $y(n)$ = Sum of the residues

$$= \frac{2^{n}\left[5n - 2^{3}\right]}{25} + \frac{i^{n-1}}{(i-2)^{2}} + \frac{(-i)^{n-1}}{(i+2)^{2}}$$
$$+ \frac{1}{2}(i)^{n-1}(Ai + B) + \frac{1}{2}A(-i)^{n} + \frac{B}{2}(-i)^{n-1},$$
$$\text{where} \quad A = y(0)\ \&\ B = y(1)$$

19. Solve the difference using Z-transform $y_{(n+3)} - 3y_{(n+1)} + 2y_{(n)} = 0$ **given that** $y_0 = 4, y_1 = 0, y_2 = 8$ [AU N/D 2012]

Solution:

Given $y(n+3) - 3y(n+1) + 2y(n) = 0$

$$Z\left[y(n+3)\right] - 3Z\left[y(n+1)\right] + 2Z\left[y(n)\right] = 0$$
$$\left[z^{3}Y(z) - z^{3}y(0) - z^{2}y(1) - zy(2)\right] - 3\left[zy(z) - zy(0)\right] + 2y(z) = 0$$
$$\left[z^{3}Y(z) - 4z^{3} - 8z\right] - 3\left[zy(z) - 4z\right] + 2y(z) = 0$$
$$\left(z^{3} - 3z + 2\right)y(z) - 4z^{3} + 4z = 0$$
$$(z-1)^{2}(z+2)y(z) = 4z^{3} - 4z$$
$$y(z) = \frac{4z^{3} - 4z}{(z-1)^{2}(z+2)}$$
$$= \frac{4z\left[z^{2} - 1\right]}{(z-1)^{2}(z+2)}$$
$$= \frac{4z[z-1][z+1]}{(z-1)^{2}(z+2)}$$
$$= \frac{4z[z+1]}{(z-1)(z+2)}$$

$$y(z)z^{n-1} = \frac{4z^n[z+1]}{(z-1)(z+2)}$$

$y(z)z^{n-1}$ has poles at z = 1 and z = -2

z = 1 is a simple pole

z = -2 is a simple pole

$$\operatorname*{Res}_{z=1} y(z)z^{n-1} = \lim_{z\to 1}(z-1)\frac{4z^n[z+1]}{(z-1)(z+2)}$$
$$= \lim_{z\to 1}\frac{4z^n[z+1]}{(z+2)}$$
$$= \frac{8}{3}$$

$$\operatorname*{Res}_{z=-2} y(z)z^{n-1} = \lim_{z\to -2}(z+2)\frac{4z^n[z+1]}{(z-1)(z+2)}$$
$$= \lim_{z\to -2}\frac{4z^n[z+1]}{(z-1)}$$
$$= \frac{4(-1)(-2)^n}{-3}$$
$$= \frac{4}{3}(-2)^n$$

Result: $y(n)$ = sum of the residues

$$= \frac{8}{3} + \frac{4}{3}(-2)^n$$

20. Solve $y_{(n+2)} + 6y_{(n+1)} + 9y_{(n)} = 2^n$ **given that** $y_0 = y_1 = 0$ **[AU N/D 2012, 2016]**

Solution:

Given $y_{(n+2)} + 6y_{(n+1)} + 9y_{(n)} = 2^n$

$$Z\left(y_{(n+2)}\right) + 6Z\left(y_{(n+1)}\right) + 9Z\left(y_{(n)}\right) = Z\left(2^n\right)$$

$$\left[z^2 y(z) - z^2 y(0) - zy(1)\right] + 6\left[zy(z) - zy(0)\right] + 9y(z) = \frac{z}{z-2}$$

$$z^2 y(z) + 6zy(z) + 9y(z) = \frac{z}{z-2}$$

$$\left[z^2 + 6z + 9\right]y(z) = \frac{z}{z-2}$$

$$(z+3)^2 y(z) = \frac{z}{z-2}$$

$$y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$y(z)z^{n-1} = \frac{z^n}{(z-2)(z+3)^2}$$

$y(z)z^{n-1}$ has poles at z=2 and z=-3

z=2 is a simple pole

z=-3 is a pole of order 2

$$\operatorname*{Res}_{z=-3} y(z)z^{n-1} = \lim_{z\to -3}\frac{1}{1!}\frac{d}{dz}\left[\frac{(z+3)^2 z^n}{(z-2)(z+3)^2}\right]$$
$$=\lim_{z\to -3}\frac{d}{dz}\left[\frac{z^n}{(z-2)}\right]$$
$$=\lim_{z\to -3}\frac{d}{dz}\left[\frac{z^n}{(z-2)}\right]$$
$$=\lim_{z\to -3}\left[\frac{(z-2)nz^{n-1}-z^n}{(z-2)^2}\right]$$
$$=\frac{(-5)n(-3)^{n-1}-(-3)^n}{(-5)^2}$$
$$=\frac{(-3)^n\left[(-5)n(-3)^{-1}-1\right]}{(-5)^2}$$
$$=\frac{(-3)^n\left[\frac{5}{3}n-1\right]}{25}$$
$$=\frac{(-3)^n[5n-3]}{75}$$

$$\operatorname*{Res}_{z=2} y(z)z^{n-1} = \lim_{z\to 2}(z-2)\left[\frac{z^n}{(z-2)(z+3)^2}\right]$$
$$=\lim_{z\to 2}\left[\frac{z^n}{(z+3)^2}\right]$$
$$=\frac{2^n}{5^2}$$
$$=\frac{2^n}{25}$$

Result: $y(n)$ = sum of the residue

$$=\frac{(-3)^n[5n-3]}{75}+\frac{2^n}{25}$$
$$=\frac{1}{25}\left[2^n-(-3)^n+\frac{5}{3}n(-3)^n\right]$$

21. Solve using Z-transform, $u_{n+2}-3u_{n+1}+2u_n=4^n$ **given** $u_0=0$ **and** $u_1=1$

Solution: **[AU A/M 2014]**

Given that

$$u_{n+2}-3u_{n+1}+2u_n=4^n$$

Taking Z-transform, we get

$$Z[u(n+2)]-3Z[u(n+1)]+2Z[u(n)]=Z(4^n)$$

$$z^2\bar{u}-z^2u(0)-zu(1)-3\left[z\bar{u}-zu(0)\right]+2\bar{u}=\frac{z}{z-4} \quad \ldots.(1) \text{ Where } \bar{u}=Z[u(n)]$$

Given $u(0)=0$, $u(1)=1$ ….(2)

Substituting (2) in(1), we get

$$z^2\left[\bar{u}-z\right]-3z\left[\bar{u}\right]+2\bar{u}=\frac{z}{z-4}$$

$$\bar{u}\left[z^2-3z+2\right]-z=\frac{z}{z-4}$$

$$\bar{u}\left[z^2-3z+2\right]=\frac{z}{z-4}+z$$

$$\bar{u}=\frac{1}{(z-1)(z-2)}\left[\frac{z}{z-4}+z\right]$$

$$=\frac{z}{(z-1)(z-2)(z-4)}+\frac{z}{(z-1)(z-2)} \qquad ----- \ (A)$$

$$\frac{\bar{u}}{z}=\frac{1}{(z-1)(z-2)(z-4)}+\frac{1}{(z-1)(z-2)} \qquad \ldots.(3)$$

Let $\dfrac{1}{(z-1)(z-2)(z-4)}=\dfrac{A}{z-1}+\dfrac{B}{z-2}+\dfrac{C}{z-4}$

$$1=A(z-2)(z-4)+B(z-1)(z-4)+C(z-1)(z-2)$$

Put $z=2$	Put $z=1$	Put $z=4$
$1=-2B$	$1=3A$	$1=6C$
$B=-\dfrac{1}{2}$	$A=\dfrac{1}{3}$	$C=\dfrac{1}{6}$

$$\frac{1}{(z-1)(z-2)(z-4)}=\frac{1}{3}.\frac{1}{z-1}-\frac{1}{2}.\frac{1}{z-2}+\frac{1}{6}.\frac{1}{z-4} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Let $\dfrac{1}{(z-1)(z-2)}=\dfrac{A}{z-1}+\dfrac{B}{z-2}$

$$1=A(z-2)+B(z-1)$$

Put $z=2$	Put $z=1$
$1=B$	$-1=A$

$$\frac{1}{(z-1)(z-2)}=-1.\frac{1}{z-1}+.\frac{1}{z-2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Sub. (4) & (5) *in* (*A*)

$$\bar{u}=\left[\frac{1}{3}.\frac{z}{z-1}-\frac{1}{2}.\frac{z}{z-2}+\frac{1}{6}.\frac{z}{z-4}\right]+\left[\frac{z}{z-2}-\frac{z}{z-1}\right]$$

$$Z[u(n)]=\left[\frac{1}{3}.\frac{z}{z-1}-\frac{1}{2}.\frac{z}{z-2}+\frac{1}{6}.\frac{z}{z-4}\right]+\left[\frac{z}{z-2}-\frac{z}{z-1}\right]$$

Result: $u(n)=Z^{-1}\left[\dfrac{1}{3}.\dfrac{z}{z-1}-\dfrac{1}{2}.\dfrac{z}{z-2}+\dfrac{1}{6}.\dfrac{z}{z-4}\right]+Z^{-1}\left[\dfrac{z}{z-2}-\dfrac{z}{z-1}\right]$

$$= \frac{1}{3}Z^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{6}Z^{-1}\left[\frac{z}{z-4}\right] + Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{z}{z-1}\right]$$

$$= \frac{1}{3}(1)^n - \frac{1}{2}(2)^n + \frac{1}{6}(4)^n + (2)^n - (1)^n \quad , n \geq 0$$

22. Solve the Equation $y_{n+2} - 3y_{n+1} + 2y_n = 0$ **given that** $y_0 = 0, y_1 = 1$

Solution: **[AU N /D 2014, A/M 2015]**

$$y(n+2) - 3y(n+1) + 2y(n) = 0 \qquad \ldots (1)$$

Taking Z-Transform on both sides, we get,

$$Z[y(n+2)] - 3Z[y(n+1)] + 2Z[y(n)] = 0$$

$$z^2\bar{y} - z^2 y(0) - zy(1) - 3z[\bar{y} - y(0)] + 2\bar{y} \qquad \ldots. (2) \qquad where \quad \bar{y} = Z[y(n)]$$

Applying y (0) =0 and y (1) =1 in (2)

$$z^2[\bar{y}] - 3z\bar{y} + 2\bar{y} = z$$

$$\bar{y} = \frac{z}{(z-1)(z-2)}$$

$$\frac{\bar{y}}{z} = \frac{1}{(z-1)(z-2)} \qquad \ldots\ldots\ldots\ldots\ldots.(1)$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

$$Put\ z = 1 \qquad Put\ z = 2$$

$$A = -1 \qquad B = 1$$

$$\frac{\bar{y}}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\bar{y} = \frac{-z}{z-1} + \frac{z}{z-2}$$

$$Z[y(n)] = \frac{-z}{z-1} + \frac{z}{z-2}$$

$$y(n) = Z^{-1}\left[\frac{-z}{z-1}\right] + Z^{-1}\left[\frac{z}{z-2}\right]$$

Result: $y(n) = (2)^n - 1 \quad , n > 1$

23. Using Z – transforms, Solve the equation $y_{n+2} - 7y_{n+1} + 12y_n = 2^n$ **given that** $y_0 = 0, y_1 = 0$ **[AU M/J 2017, N/D 2017]**

Solution:

$$y(n+2) - 7y(n+1) + 12y(n) = 2^n \qquad \ldots (1)$$

Taking Z-Transform on both sides, we get,

$$Z[y(n+2)] - 7Z[y(n+1)] + 12Z[y(n)] = Z[2^n]$$

$$z^2\bar{y} - z^2 y(0) - zy(1) - 7z[\bar{y} - y(0)] + 2\bar{y} = \frac{z}{z-2} \qquad \ldots. (2) \qquad where \quad \bar{y} = Z[y(n)]$$

Applying y (0) =0 and y (1) =1 in (2)

$$z^2\left[\bar{y}\right]-7z\bar{y}+12\bar{y}=\frac{z}{z-2}$$

$$\bar{y}=\frac{z}{(z-2)(z-3)(z-4)}$$

$$\frac{\bar{y}}{z}=\frac{1}{(z-2)(z-3)(z-4)} \quad \ldots\ldots\ldots\ldots\ldots.(3)$$

$$\frac{1}{(z-2)(z-3)(z-4)}=\frac{A}{z-2}+\frac{B}{z-3}+\frac{C}{z-4}$$

$$1=A(z-3)(z-4)+B(z-2)(z-4)+C(z-2)(z-3)$$

$$Put \quad z=2\Rightarrow A=\frac{1}{2}, \quad Put \; z=3\Rightarrow B=-1, \quad Put \; z=1\Rightarrow C=\frac{1}{2}$$

$$\frac{\bar{y}}{z}=\frac{\frac{1}{2}}{z-2}-\frac{1}{z-3}+\frac{\frac{1}{2}}{z-4}$$

$$\bar{y}=\frac{1}{2}\frac{z}{z-2}-\frac{z}{z-3}+\frac{1}{2}\frac{z}{z-4}$$

$$Z\left[y(n)\right]=\frac{1}{2}\frac{z}{z-2}-\frac{z}{z-3}+\frac{1}{2}\frac{z}{z-4}$$

$$y(n)=\frac{1}{2}Z^{-1}\left(\frac{z}{z-2}\right)-Z^{-1}\left(\frac{z}{z-3}\right)+\frac{1}{2}Z^{-1}\left(\frac{z}{z-4}\right)$$

$$y(n)=\frac{1}{2}(2)^n-(3)^n+\frac{1}{2}(4)^n$$

Result: $$y(n)=\frac{1}{2}(2)^n-(3)^n+\frac{1}{2}(4)^n$$

PROBLEMS BASED ON FORMATION METHOD

24. Form the difference equation from $y_n = A2^n + Bn$

Solution:

$$y_n = A2^n + Bn$$

$$y_{n+1} = A2^{n+1} + B(n+1) \quad \ldots.\ (1)$$

$$= 2A2^n + B(n+1) \quad \ldots.\ (2)$$

$$y_{n+2} = A2^{n+2} + B(n+2)$$

$$y_{n+2} = 4A2^n + B(n+2) \quad \ldots.\ (3)$$

Eliminating A and B, we get

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & n+1 \\ y_{n+2} & 4 & n+2 \end{vmatrix} = 0$$

$$y_n\left[2(n+2)-4(n+1)\right]-1\left[(n+2)y_{n+1}-(n+1)y_{n+2}\right]+n\left[4y_{n+1}-2y_{n+2}\right]=0$$

$$y_n(-2n)+(3n-2)y_{n+1}+(1-n)y_{n+2}=0$$

Result: $$(1-n)y_{n+2}+(3n-2)y_{n+1}-2ny_n=0$$

25. Form the difference equation by eliminating the constants from $y_n = a2^n + b(-2)^n$

Solution:

$$y_n = a2^n + b(-2)^n \qquad \text{…. (1)}$$

$$y_{n+1} = a2^{n+1} + b(-2)^{n+1}$$

$$y_n = 2a2^n - 2b(-2)^n \qquad \text{…. (2)}$$

$$y_{n+2} = a2^{n+2} + b(-2)^{n+2}$$

$$y_{n+2} = 4a2^n + 4b(-2)^n \qquad \text{…. (3)}$$

Eliminating 'a' and 'b' from (1), (2) and (3), we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0$$

$$y_n(8+8) - 4(y_{n+1} + 2y_{n+2}) + 1(4y_{n+1} - 2y_{n+2}) = 0$$

$$16y_n - 4y_{n+2} = 0$$

Result: $y_{n+2} - 4y_n = 0$

26. Form the difference equation by eliminating the constants from $y_n = [A + Bn]2^n$ **[AU N/D 2013]**

Solution:

$$y_n = [A + Bn]2^n$$

$$= A2^n + Bn2^n \qquad \text{…. (1)}$$

$$y_{n+1} = A2^{n+1} + B(n+1)(2)^{n+1}$$

$$= 2A2^n + 2B(n+1)(2)^n \qquad \text{…. (2)}$$

$$y_{n+2} = A2^{n+2} + B(n+2)(2)^{n+2}$$

$$= 4A2^n + 4B(n+2)(2)^n \qquad \text{…. (3)}$$

Eliminating 'A' and '$B2^n$' from (1), (2) and (3), we get

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$y_n(8(n+2) - 8(n+1)) - 1(4(n+2)y_{n+1} - 2(n+1)y_{n+2}) + n(4y_{n+1} - 2y_{n+2}) = 0$$

$$8y_n - (4n+8)4y_{n+1} + (2n+2)y_{n+2} + 4ny_{n+1} - 2ny_{n+2} = 0$$

$$2y_{n+2} - 8y_{n+1} + 8y_n = 0$$

Result: $y_{n+2} - 4y_{n+1} + 4y_n = 0$

27. Find the Z- transform of $\dfrac{1}{(n+1)(n+2)}$ **[AU N/D 2013]**

Solution:

Consider $\dfrac{1}{(n+1)(n+2)} = \dfrac{A}{n+1} + \dfrac{B}{n+2}$

$$1 = A(n+2) + B(n+1)$$

Let n= -2 then B= -1

n= -1 then A=1

$$z\left[\frac{1}{(n+1)(n+2)}\right] = z\left[\frac{1}{(n+1)}\right] + z\left[\frac{-1}{(n+2)}\right]$$

$$wkt \ Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{n+2} z^{-n}$$

$$= 1 + \frac{z^{-1}}{2} + \frac{z^{-2}}{3} + ... + \frac{1}{2} + \frac{z^{-1}}{3} + \frac{z^{-2}}{4} + ...$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + ... + \frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + ...$$

$$= z\left(\frac{1}{z}\right)\left[1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + ...\right] + z^2\left[\frac{1}{z^2}\right]\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + ...\right]$$

$$= z\left(\frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + ..\right) + z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + ...\right]$$

Result: $= z\log\left(\frac{z}{z-1}\right) + z^2 \log\frac{z}{z-1} - z$

PROBLEMS BASED ON RESIDUE METHOD

28. Find $z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$ **[AU N/D 2007, A/M 2000, 2005]**

Solution:

$$F(Z) = \frac{Z(Z+1)}{(Z-1)^3} \quad , \ Z^{-1}[F(z)] = f(n)$$

$$\therefore z^{n-1}F(z) = \frac{z^n(z+1)}{(z-1)^3}$$

z =1 is a pole of order 3

$$\operatorname{Re}s\{z^{n-1}F(Z)\}_{Z=1} = \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3 . \frac{z^n(z+1)}{(z-1)^3}\right]$$

$$= \lim_{z\to 1}\frac{1}{2}\frac{d^2}{dz^2}\left[z^n(z+1)\right]$$

$$= \lim_{z\to 1}\frac{1}{2}\frac{d}{dz}\left[z^n + (z+1)nz^{n-1}\right]$$

$$= \lim_{z\to 1}\frac{1}{2}\left[nz^{n-1} + n\{(z+1)(n-1)z^{n-2} + z^{n-1}\}\right]$$

$$= \frac{1}{2}\left[n(1)^{n-1} + n\left\{2(n-1)(1)^{n-2} + (1)^{n-1}\right\}\right]$$
$$= \frac{1}{2}\left[n(1)^{n-1} + 2n(n-1)(1)^{n-2} + n(1)^{n-1}\right]$$
$$= \frac{1}{2}\left[2n(1)^{n-1} + 2n(n-1)(1)^{n-2}\right]$$
$$= n(1)^{n-1} + n(n-1)(1)^{n-2}$$
$$= n + n(n-1)$$
$$= n + n^2 - n$$
$$= n^2$$

Result:
$$f(n) = Sum\ of\ the\ residues$$
$$= n^2$$

29. Find $Z^{-1}\left[\dfrac{z}{(z-1)(z^2+1)}\right]$

Solution: **[AU A/M 2007, 2008 N/D 2000]**

Given that

$$\frac{z}{(z-1)(z^2+1)} = F(z) \quad , \quad Z^{-1}[F(z)] = f(n)$$

$$z^{n-1}F(Z) = \frac{z^n}{(z-1)(z^2+1)}$$

The poles are z=1, z=+i, z=-i (simple pole)

$$\operatorname{Res}\{z^{n-1}F(Z)\}_{Z=1} = \lim_{z\to 1}(z-1)\frac{z^n}{(z-1)(z^2+1)}$$
$$= \frac{(1)^n}{2}$$

$$\operatorname{Res}\{z^{n-1}F(Z)\}_{Z=i} = \lim_{z\to i}(z-i)\frac{z^n}{(z-1)(z+i)(z-i)}$$
$$= \frac{(i)^n}{(i-1)(2i)}$$

$$\operatorname{Res}\{z^{n-1}F(Z)\}_{Z=-i} = \lim_{z\to -i}(z+i)\frac{z^n}{(z-1)(z+i)(z-i)}$$
$$= \frac{(-i)^n}{(-i-1)(-2i)}$$

f(n) =sum of the residue of z^{n-1} F(z)

$$= \frac{(1)^n}{2} + \frac{(i)^n}{(i-1)(2i)} + \frac{(-i)^n}{(-i-1)(-2i)}$$
$$= \frac{1}{2}\left[(-1)^n - \frac{(i)^n}{(1-i)(i)} + \frac{(-i)^n}{(1+i)(i)}\right]$$

$$=\frac{1}{2}\left[\frac{2i(1)^n-(i)^n(1+i)+(-i)^n(1-i)}{i(1+i)(1-i)}\right]$$

$$=\frac{1}{2}\left[\frac{2i-(1+i)\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right)+(1-i)\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)}{2i}\right]$$

$$=\frac{1}{4i}\left[2i-\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}-i\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}-i\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right]$$

$$=\frac{1}{4i}\left[2i-2i\cos\frac{n\pi}{2}-2i\sin\frac{n\pi}{2}\right]$$

Result:

$$f(n)=\frac{1}{2}\left[1-\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right]$$

30. Find $Z^{-1}\left[\frac{z^2}{z^2+4}\right]$ **[AU N/D 2010, A/M 2003,2004]**

Solution:

$$F(Z)=\left[\frac{z^2}{z^2+4}\right] \quad , \quad Z^{-1}[F(z)]=f(n)$$

$$\therefore Z^{n-1}F(Z)=\frac{z^{n+1}}{z^2+4}=\frac{z^{n+1}}{(z+2i)(z-2i)}$$

The poles are z=2i, z=-2i

$$\operatorname{Res}\{z^{n-1}F(Z)\}_{Z=2i}=\lim_{z\to 2i}(z-2i)\frac{z^{n+1}}{(z+2i)(z-2i)}$$

$$=\frac{(2i)^{n+1}}{4i}$$

$$=\frac{(2i)^n}{2}$$

$$=(2)^{n-1}(i)^n$$

$$\operatorname{Res}\{z^{n-1}F(Z)\}_{Z=-2i}=\lim_{z\to -2i}(z+2i)\frac{z^{n+1}}{(z+2i)(z-2i)}$$

$$=\frac{(-2i)^{n+1}}{(-4i)}$$

$$=\frac{(-2i)(-2i)^n}{(-4i)}$$

$$=\frac{1}{2}(2)^n(-i)^n$$

$$=(2)^{n-1}(-i)^n$$

Result:

$\therefore f(n)=$ {Sum of the residue of z^{n-1} F (z) at its Poles}

$$= (2)^{n-1}(i)^n + (2)^{n-1}(-i)^n$$

$$= 2^{n-1}\left[\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right] \quad \left[\because (i)^n = \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right]$$

$$= 2^n.\cos\frac{n\pi}{2}$$

31. Using residue method, find $Z^{-1}\left[\dfrac{z}{(z^2-2z+2)}\right]$

Solution: **[AU A/M 2007, '08, '15 '16]**

$$Let \quad \frac{z}{(z^2-2z+2)} = F(z) \qquad , \qquad Z^{-1}[F(z)] = f(n)$$

$$z^{n-1}F(Z) = \frac{z}{(z^2-2z+2)} z^{n-1}$$

The poles are z=1+i, z=1--i (simple poles)

$$\operatorname{Re} s\{z^{n-1}F(Z)\}_{Z=1+i} = \lim_{z\to 1+i}[z-(1+i)]\frac{z^n}{[z-(1+i)][(z-(1-i)]}$$

$$= \frac{(1+i)^n}{2i}$$

$$\operatorname{Re} s\{z^{n-1}F(Z)\}_{Z=1-i} = \lim_{z\to 1-i}[z-(1-i)]\frac{z^n}{[z-(1+i)][(z-(1-i)]}$$

$$= \frac{(1-i)^n}{-2i}$$

Result:

f(n) = Sum of the residue of z^{n-1} F(z)

$$= \left[\frac{(1+i)^n}{2i} - \frac{(1-i)^n}{2i}\right]$$

32. Using the inversion integral method (Residue Theorem), find the inverse Z-transform of $U(z) = \dfrac{z^2}{(z+2)(z^2+4)}$ **[AU N/D 2015]**

Solution:

$$U(Z) = \left[\frac{z^2}{(z+2)(z^2+4)}\right] \qquad , Z^{-1}[U(z)] = u(n)$$

$$\therefore Z^{n-1}U(Z) = \frac{z^{n+1}}{(z+2)(z^2+4)} = \frac{z^{n+1}}{(z+2)(z+2i)(z-2i)}$$

The poles are z=-2, z=2i, z=-2i

$$\operatorname{Re} s\{z^{n-1}U(z)\}_{Z=-2} = \lim_{z\to -2}(z+2)\frac{z^{n+1}}{(z+2)(z+2i)(z-2i)}$$

$$= \frac{(-2)^{n+1}}{(-2+2i)(-2-2i)}$$

$$= \frac{(-2)^{n+1}}{(4+4)}$$

$$= \frac{(-2)^{n+1}}{8}$$

$$\operatorname{Re}s\{z^{n-1}U(z)\}_{Z=2i} = \lim_{z\to 2i}(z-2i)\frac{z^{n+1}}{(z+2)(z+2i)(z-2i)}$$

$$= \frac{(2i)^{n+1}}{(2+2i)4i}$$

$$= \frac{(2i)^{n+1}}{8i(1+i)}$$

$$= \frac{(2i)^{n}}{4(1+i)}$$

$$\operatorname{Re}s\{z^{n-1}U(z)\}_{Z=-2i} = \lim_{z\to -2i}(z+2i)\frac{z^{n+1}}{(z+2)(z+2i)(z-2i)}$$

$$= \frac{(-2i)^{n+1}}{(-2i+2)(-4i)}$$

$$= \frac{(-2i)^{n+1}}{(-8i)(1-i)}$$

$$= \frac{(-2i)^{n}}{4(1-i)}$$

Result:

$\therefore u(n) = \{$Sum of the residue of z^{n-1} F (z) at its Poles$\}$

$$= \frac{(-2)^{n+1}}{8} + \frac{(2i)^{n+1}}{4(1+i)} + \frac{(-2i)^{n+1}}{4(1-i)}$$

33. Evaluate $Z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right]$, using calculus of residues. [AU N/D 2016]

Solution:

Let $F(Z) = \left[\frac{9z^3}{(3z-1)^2(z-2)}\right]$, $Z^{-1}[F(z)] = f(n)$

$$\therefore z^{n-1}F(z) = \left[\frac{9z^{n+2}}{(3z-1)^2(z-2)}\right]$$

The poles of $z^{n-1}F(z)$ are $(3z-1)^2(z-2) = 0$

$z = 2$ *is a simple pole*

$z = \frac{1}{3}$ *is a pole of order* 2

$$\operatorname{Re}s\{z^{n-1}F(Z)\}_{Z=2} = \lim_{z\to 2}(z-2)\left[\frac{9z^{n+2}}{(3z-1)^2(z-2)}\right]$$

$$= \frac{9.2^{n+2}}{25}$$

$$\operatorname{Res}\{z^{n-1}F(Z)\}_{Z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}} \frac{1}{1!}\frac{d}{dz}\left[\left(z-\frac{1}{3}\right)^2 \frac{9z^{n+2}}{(3z-1)^2(z-2)}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{z^{n+2}}{z-2}\right]$$

$$= \lim_{z\to\frac{1}{3}} \left[\frac{(z-2)(n+2)z^{n+1} - z^{n+2}}{(z-2)^2}\right]$$

$$= \left[\frac{\left(\frac{-5}{3}\right)(n+2)\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{3}\right)^{n+2}}{\frac{25}{9}}\right]$$

$$= -\left(\frac{1}{3}\right)^n \left[\frac{\left(\frac{5}{3}\right)(n+2)\left(\frac{1}{3}\right) + \left(\frac{1}{9}\right)}{\frac{25}{9}}\right]$$

$$= -\left(\frac{1}{3}\right)^n \left(\frac{1}{25}\right)[5n+11]$$

$$f(n) = \frac{9.2^{n+2}}{25} - \left(\frac{1}{3}\right)^n \left(\frac{1}{25}\right)[5n+11]$$

34. State and Prove the Final Value Theorem.

(OR)

If $Z[f(n)] = F(z)$ **then** $\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(Z)$ **[AU A/M 2007]**

Proof:

$$Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1)z^{-n}$$

$$Put\ n+1 = m$$

$$Z[f(m)] = \sum_{m=1}^{\infty}\{f(m)\}z^{-m+1}$$

$$= z[F(z) - f(0)]$$

$$Let\ \ zF(Z) - zf(0) - F(z) = z[f(n+1)] - zf(n)$$

$$(z-1)F(z) - zf(0) = \sum_{n=0}^{\infty}\{f(n+1) - f(n)\}z^{-n}$$

$$Taking\ \ \lim its\ as\ \ z\to 1$$

$$\lim_{z\to 1}(z-1)F(z) - f(0) = \sum_{n=0}^{\infty}\{f(n+1) - f(n)\}$$

$$= \lim_{n\to\infty}\left\{\begin{array}{l}[f(1) - f(0)] + [f(2) - f(1)] + \\ \qquad [f(3) - f(2)] + \ldots + [f(n+1) - f(n)]\end{array}\right\}$$

$$= \lim_{n \to \infty} \left[f(n+1) - f(0) \right]$$

$$= \lim_{n \to \infty} f(n) - f(0)$$

$$\lim_{z \to 1} (z-1)F(z) = \lim_{n \to \infty} f(n)$$

$$\lim_{z \to 1} (z-1)F(z) = \lim_{n \to \infty} f(n)$$

35. State and Prove the Second Shifting Theorem. **[AU A/M 2013]**

(OR)

If $Z\left[f(t)\right] = F(z)$ **then prove that** $Z\left[f(t+T)\right] = zF(z) - zf(0)$

Proof:

$$z\left[f(t+T)\right] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n+1)Tz.z^{-1}z^{-n}$$

$$= z\sum_{n=0}^{\infty} f(n+1)Tz^{-(n+1)}$$

Put n+1=m n=0→m=1

n=∞→m=∞

$$Z\left[f(n+1)\right] = z\sum_{m=1}^{\infty} f(mT)z^{-m}$$

$$= z\left[\sum_{m=0}^{\infty} f(mT)z^{-m} - f(0)\right]$$

$$= zF(z) - zf(0).$$

36. Find $Z[\cos n\theta]$ **and** $Z[\sin n\theta]$.

Solution: **[AU N/D 2002, M/J 2005, '08, '13, '16]**

We know that, $Z\left[a^n\right] = \dfrac{z}{z-a}$

$$put \quad a = e^{i\theta}$$

$$\therefore Z\left[\left(e^{i\theta}\right)^n\right] = \frac{z}{z - e^{i\theta}}$$

$$Z\left[\left(e^{in\theta}\right)\right] = \frac{z}{z - (\cos\theta + i\sin\theta)}$$

$$Z\left[\cos n\theta + i\sin n\theta\right] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta} \times \frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta)+iz\sin\theta}{z^2+\cos^2\theta-2z\cos\theta+\sin^2\theta}$$

$$= \frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z[\cos n\theta]+iZ[\sin n\theta]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}+\frac{iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equating real and Imaginary parts, we get

$$Z[\cos n\theta]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

Put $\theta=\frac{\pi}{2}$

Result:

$$Z\left[\cos n\frac{\pi}{2}\right]=\frac{z^2}{z^2+1}$$

$$Z[\sin n\theta]=\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Put $\theta=\frac{\pi}{2}$

$$Z\left[\sin n\frac{\pi}{2}\right]=\frac{z}{z^2+1}$$

37. Find the Z-transform of $\sin^2\left(\frac{n\pi}{4}\right)$ and $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$. [AU N/D 2012]

Solution:

To find: $Z\left[\sin^2\left(\frac{n\pi}{4}\right)\right]$

We know that $\sin^2\theta=\frac{1-\cos 2\theta}{2}$

$$\sin^2\left(\frac{n\pi}{4}\right)=\frac{1-\cos\frac{n\pi}{2}}{2}$$

$$=\frac{1}{2}-\frac{1}{2}\cos\frac{n\pi}{2}$$

$$Z\left[\sin^2\left(\frac{n\pi}{4}\right)\right]=\frac{1}{2}Z(1)-\frac{1}{2}Z\left[\cos\frac{n\pi}{2}\right]$$

$$=\frac{1}{2}\left[\frac{z}{z-1}\right]-\frac{1}{2}\left[\frac{z^2}{z^2+1}\right]$$

$$=\frac{1}{2}\left[\frac{z}{z-1}-\frac{z^2}{z^2+1}\right]$$

We know that

$$Z[\cos n\theta]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

Put $\theta=\frac{\pi}{2}$ we get

$$Z\left[\cos\frac{n\pi}{2}\right]=\frac{z^2}{z^2+1}$$

To find: $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]$

We know that , $\cos(A+B)=\cos A\cos B-\sin A\sin B$

$$\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)=\cos\frac{n\pi}{2}\cos\frac{\pi}{4}-\sin\frac{n\pi}{2}\sin\frac{\pi}{4}$$

$$= \cos\frac{n\pi}{2}\frac{1}{\sqrt{2}} - \sin\frac{n\pi}{2}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = \frac{1}{\sqrt{2}}Z\left[\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}Z\left[\cos\frac{n\pi}{2}\right] - \frac{1}{\sqrt{2}}Z\left[\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{\sqrt{2}}\frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}}\frac{z}{z^2+1}$$

$$= \frac{z}{\sqrt{2}}\left[\frac{z}{z^2+1} - \frac{1}{z^2+1}\right]$$

$$= \frac{z}{\sqrt{2}}\left[\frac{z-1}{z^2+1}\right]$$

Result:

$$= \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

38. Find the Z-transforms of $r^n \cos n\theta$ and $e^{-at}\cos bt$

Solution: **[AU A/M 2014]**

(i) We know that $Z[a^n] = \dfrac{z}{z-a}$

Put $a = r\,e^{i\theta}$

$$Z\left[\left(r\,e^{i\theta}\right)^n\right] = \frac{z}{z - r\,e^{i\theta}}$$

$$Z\left[\left(r^n\,e^{in\theta}\right)\right] = \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$Z\left[r^n(\cos n\theta + i\sin n\theta)\right] = \frac{z}{z - r\cos\theta - ir\sin\theta}$$

$$= \frac{z}{z - r\cos\theta - ir\sin\theta}\; x\, \frac{(z - r\cos\theta) + ir\sin\theta}{(z - r\cos\theta) + ir\sin\theta}$$

$$= \frac{z(z - r\cos\theta) + irz\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + irz\sin\theta}{z^2 + r^2\cos^2\theta - 2zr\cos\theta + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + irz\sin\theta}{z^2 - 2zr\cos\theta + r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{z(z - r\cos\theta) + irz\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

$$Z\left[r^n\cos n\theta + i\,r^n\sin n\theta\right] = \frac{z(z - r\cos\theta) + irz\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

Equating real and imaginary parts, we have

$$Z\left[r^n \cos n\theta\right] = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} \quad , \quad Z\left[r^n \sin n\theta\right] = \frac{rz\sin\theta}{z^2 - 2zr\cos\theta + r^2}$$

(ii) $Z\left[e^{-at}\ f(t)\right] = \{F(z)\}_{z \to z\,e^{aT}}$

Here $f(t) = \cos bt$

$$Z\left[e^{-at}\cos bt\right] = \{Z(\cos bt)\}_{z \to z\,e^{aT}} \qquad \text{(By shifting property)}$$

$$= \left\{\frac{z(z - \cos bT)}{z^2 - 2z\cos bT + 1}\right\}_{z \to z\,e^{at}}$$

Result:

$$= \frac{z\,e^{aT}(z\,e^{aT} - \cos bT)}{z^2\,e^{2aT} - 2ze^{aT}\cos bT + 1}$$

39. Find Z – Transform of $\frac{2n+3}{(n+1)(n+2)}$ **[AU M/J 2017, N/D 2017]**

Solution:

Let $f(n) = \frac{2n+3}{(n+1)(n+2)}$

Let $\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$

$$2n+3 = A(n+2) + B(n+1)$$

Put $n = -1,\ A = 1$

$n = -2,\ B = 1$

$$f(n) = \frac{1}{n+1} + \frac{1}{n+2}$$

$$Z\{f(n)\} = Z\left\{\frac{1}{n+1}\right\} + Z\left\{\frac{1}{n+2}\right\} \text{..................(1)}$$

$$Z\left\{\frac{1}{n+1}\right\} = z\log\left(\frac{z}{z-1}\right)$$

$$Z\left\{\frac{1}{n+2}\right\} = \sum_{n=0}^{\infty}\frac{1}{n+2}z^{-n} = \frac{1}{2} + \frac{1}{3}\frac{1}{z} + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \text{........}$$

$$= z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + \text{........}\right]$$

$$= z^2\left[-\log\left(1 - \frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= z^2\log\left(\frac{z}{z-1}\right) - z$$

Substitute in (1), we get

$$Z\{f(n)\} = z\log\left(\frac{z}{z-1}\right) + z^2\log\left(\frac{z}{z-1}\right) - z$$

$$= z(z+1)\log\left(\frac{z}{z-1}\right) - z$$

40. Find $Z^{-1}\left[\frac{4z^3}{(2z-1)^2(z-1)}\right]$ **by the method of partial fractions. [AU M/J 2017]**

Solution:

$$Let \quad F(z) = \left[\frac{4z^3}{(2z-1)^2(z-1)}\right]$$

$$\frac{F(z)}{z} = \left[\frac{4z^2}{(2z-1)^2(z-1)}\right]$$

$$\frac{4z^2}{(2z-1)^2(z-1)} = \frac{A}{(z-1)} + \frac{B}{(2z-1)} + \frac{C}{(2z-1)^2}$$

$$4z^2 = A(2z-1)^2 + B(2z-1)(z-1) + C(z-1)$$

$$put \;\; z = 1 \Rightarrow A = 4, \quad z = \frac{1}{2} \Rightarrow C = -2, \quad z = 0 \Rightarrow B = -6$$

$$\frac{F(z)}{z} = \frac{4}{(z-1)} - \frac{6}{(2z-1)} - \frac{2}{(2z-1)^2}$$

$$F(z) = \frac{4z}{(z-1)} - \frac{3z}{\left(z-\frac{1}{2}\right)} - \frac{1}{2}\frac{z}{\left(z-\frac{1}{2}\right)^2}$$

$$Z[f(n)] = \frac{4z}{(z-1)} - \frac{3z}{\left(z-\frac{1}{2}\right)} - \frac{\frac{1}{2}z}{\left(z-\frac{1}{2}\right)^2}$$

$$[f(n)] = Z^{-1}\left(\frac{4z}{(z-1)}\right) - Z^{-1}\left(\frac{3z}{\left(z-\frac{1}{2}\right)}\right) - Z^{-1}\left(\frac{\frac{1}{2}z}{\left(z-\frac{1}{2}\right)^2}\right)$$

$$f(n) = 4 - 3\left(\frac{1}{2}\right)^n - n\left(\frac{1}{2}\right)^n$$

41. Find $Z^{-1}\left[\frac{(z^2+z)}{(z-1)(z^2+1)}\right]$

Solution: **[AU N/D 2014]**

$$Let \qquad \frac{z^2+z}{(z-1)(z^2+1)} = F(z)$$

$$\frac{(z+1)}{(z-1)(z^2+1)} = \frac{F(z)}{z}$$

$$\frac{(z+1)}{(z^2+1)(z-1)} = \frac{A}{(z-1)} + \frac{Bz+C}{(z^2+1)}$$

$$(z+1) = A(z^2+1) + (Bz+C)(z+1)$$

$Put\ z=1,\ then\ A=1$

$Comparing\ the\ coefficient\ of\ z^2$

$$0 = A + B$$

$$B = -1$$

$Comparing\ the\ coefficient\ of\ z$

$$1 = -B + C$$

$$C = 0$$

$$\frac{F(z)}{z} = \frac{1}{(z-1)} - \frac{z}{(z^2+1)}$$

$$F(Z) = \frac{z}{(z-1)} - \frac{z^2}{(z^2+1)}$$

Result:

$$Z^{-1}[F(Z)] = Z^{-1}\left[\frac{z}{(z-1)}\right] - Z^{-1}\left[\frac{z^2}{(z^2+1)}\right]$$

$$= 1^n - \cos\frac{n\pi}{2}, n \geq 0$$

42. If $U(z) = \dfrac{z^3+z}{(z-1)^3}$**, find the value of** $u_0, u_1\ and\ u_2$

[AU N/D 2015]

Solution:

$$U(Z) = \frac{Z(Z^2+1)}{(Z-1)^3} \qquad , \ Z^{-1}[F(z)] = f(n)$$

$$\therefore z^{n-1}U(z) = \frac{z^n(z^2+1)}{(z-1)^3}$$

z =1 is a pole of order 3

$$\operatorname{Res}\{z^{n-1}U(Z)\}_{Z=1} = \lim_{z\to 1}\frac{1}{2!}\frac{d^2}{dz^2}\left[(z-1)^3.\frac{z^n(z^2+1)}{(z-1)^3}\right]$$

$$= \lim_{z\to 1}\frac{1}{2}\frac{d^2}{dz^2}\left[z^n(z^2+1)\right]$$

$$= \lim_{z\to 1}\frac{1}{2}\frac{d}{dz}\left[z^n(2z) + (z^2+1)nz^{n-1}\right]$$

$$= \lim_{z\to 1}\frac{1}{2}\left[2z^n + 2z.nz^{n-1} + nz^{n-1}.2z + n(z^2+1)(n-1)z^{n-2}\right]$$

$$= \frac{1}{2}\left[2 + 2n + 2n + 2n(n-1)\right]$$

$$= \frac{1}{2}\left[2 + 4n + 2n^2 - 2n\right]$$

$$= \frac{1}{2}\left[2 + 2n + 2n^2\right]$$

$$= 1 + n + n^2$$

Result:

$$u(n) = Sum\ of\ \ the\ residues$$
$$u(n) = 1 + n + n^2$$
$$u(0) = 1$$
$$u(1) = 3$$
$$u(2) = 7$$

43.Find the inverse Z- transform of $\dfrac{z^2 - 3z}{(z-5)(z+2)}$ using residue theorem.

[AU A/M 2018]

Solution:

Let $f(z) = \left[\dfrac{z^2 - 3z}{(z-5)(z+2)}\right]$

$$\therefore Z^{n-1}\,\mathrm{f(z)} = \frac{z^n(z-3)}{(z-5)(z+2)}$$

Z = 5, Z = -2 are simple pole.

$$\operatorname{Re}s\left\{z^{n-1}\,\mathrm{f(z)}\right\}_{Z=5} = \lim_{z\to 5}(z-5)z^{n-1}f(z)$$
$$= \lim_{z\to 5}(z-5)\frac{z^n(z-3)}{(z-5)(z+2)}$$
$$= \frac{2(5)^n}{7}$$
$$\operatorname{Re}s\left\{z^{n-1}\,\mathrm{f(z)}\right\}_{Z=-2} = \lim_{z\to -2}(z+2)z^{n-1}f(z)$$
$$= \lim_{z\to -2}(z+2)\frac{z^n(z-3)}{(z-5)(z+2)}$$
$$= \frac{5(-2)^n}{7}$$

Result:

$\therefore z^{-1}\left[f(z)\right] = \{$Sum of the residue of z^{n-1} f (z) at its Poles$\}$

$$z^{-1}\left[\frac{z^2 - 3z}{(z-5)(z+2)}\right] = \frac{2(5)^n}{7} + \frac{5(-2)^n}{7}$$

44. Find the inverse Z –transform of $\dfrac{z^3}{(z-1)^2(z-2)}$ by method of partial fractions.

[AU N/D 2017]

Solution:

$$Let\quad F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\frac{F(z)}{z}=\left[\frac{z^2}{(z-1)^2(z-2)}\right]$$

$$\frac{z^2}{(z-1)^2(z-2)}=\frac{A}{(z-1)}+\frac{B}{(z-1)^2}+\frac{C}{(z-2)}$$

$$z^2=A(z-1)(z-2)\ +B(z-2)+C(z-1)^2$$

$$put\ \ z=1\Rightarrow B=-1,\quad z=2\Rightarrow C=4,\quad z=0\Rightarrow A=-3$$

$$\frac{F(z)}{z}=\frac{-3}{(z-1)}-\frac{1}{(z-1)^2}-\frac{4}{(z-2)}$$

$$F(z)=\frac{-3z}{(z-1)}-\frac{z}{(z-1)^2}-\frac{4z}{(z-2)}$$

$$Z[f(n)]=\frac{-3z}{(z-1)}-\frac{z}{(z-1)^2}-\frac{4z}{(z-2)}$$

$$f(n)=-3Z^{-1}\left(\frac{z}{(z-1)}\right)-Z^{-1}\left(\frac{z}{(z-1)^2}\right)+4Z^{-1}\left(\frac{z}{(z-2)}\right)$$

$$f(n)=-3(1)^n-n+4(2)^n$$

ANNA UNIVERSITY IMPORTANT QUESTIONS

1. Using Convolution Theorem, Find the $Z^{-1}\left[\dfrac{z^2}{(z-4)(z-3)}\right]$

[AU N/D 2010,2015] Refer Page No: 18

2. Using Convolution theorem, find the value of $Z^{-1}\left[\dfrac{z^3}{(z-2)^2(z-3)}\right]$

[AU N/D 2010] Refer Page No: 18

3. Using Convolution theorem finds the inverse z- transform $\left[\dfrac{14z^2}{(7z-1)(2z-1)}\right]$

Refer Page No:19

4. Using Convolution Theorem Find the value of $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\right]$

[AU A/M 2012] Refer Page No:20

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5. Using Convolution theorem, Find $Z^{-1}\left[\dfrac{z^2}{(z-a)^2}\right]$ [AU N/D 2007, M/J 2016]

(or)

Find $Z^{-1}\left[(1-az^{-1})^{-2}\right]$

Refer Page No: 21

6. Using convolution theorem find the inverse Z- transform $\left[\dfrac{z^2}{(z-1)(z-3)}\right]$

[AU N/D 2006, 2013] Refer Page No: 22

7. Find $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]$ Using Convolution Theorem. [AU A/M 2010]

Refer Page No:22

8. Using Convolution Theorem, Find the $z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$

Refer Page No:23 [AU A/M 2013, 2014]

9. Solve $y_{n+2}+4y_{n+1}+3y_n=2^n$ with $y_0=0$ and $y_1=1$ using Z-transform

[AU N/D 2010, A/M 2005,2006] Refer Page No:24

10. Solve the Equation $y_{n+2}-3y_{n+1}+2y_n=2^n$ given that $y_0=y_1=0$

[AU A/M 2008, 2009, 2011] Refer Page No:25

11. Using Z-transform solve $U(n+2)-5U(n+1)+6U(n)=4^n$ given that $U(0)=0, U(1)=1$ [AU A/M 2005 N/D 2009] Refer Page No:26

12. Solve by using Z –transform $y_{n+2}+4y_{n+1}+3y_n=3^n$, given that $y_0=0$; $y_1=1$. [AU A/M 2009, 2010 N/D 2015] Refer Page No:27

13. Solve the differential equation $y(k+2)-4y(k+1)+4y(k)=0$ where $\mathbf{y(0)=1},\ \mathbf{y(1)=0}$ [AU N/D 2005, 2009, A/M 2009] Refer Page No:28

14. Solve the difference equation $y(n)+3y(n-1)-4y(n-2)=0$, $n\ge 2$, given that $\mathbf{y(0)=3}$ **and** $\mathbf{y(1)=2}$. [AU A/M 2006] Refer Page No:29

15. Solve using Z-transform, $y_{n+2}-3y_{n+1}-10y_n=0$ given $y_0=1$ and $y_1=0$ [AU N/D 2002,'08,'09, A/M 2013, 2014] Refer Page No:30

16. Solve the difference equation $y(k+2)+y(k)=1$, $y(0)=y(1)=0$, using Z-transform [AU A/M 2012] Refer Page No:31

17. Solve the difference equation $y_{n+2}+y_n=2$, $y_0=y_1=0$, using Z-transform. [AU M/J 2016] Refer Page No:32

18. Solve $y_{n+2}+y_n=n.2^n$,using Z-transform [AU A/M 2012] Refer Page No:33

19. Solve the difference using Z-transform $y_{(n+3)}-3y_{(n+1)}+2y_{(n)}=0$ given that $y_0=0, y_1=0, y_2=8$ [AU N/D 2012] Refer Page No:35

20. Solve $y_{(n+2)}+6y_{(n+1)}+9y_{(n)}=2^n$ given that $y_0=y_1=0$ [AU N/D 2012] Refer Page No:36

21. Solve using Z-transform, $u_{n+2}-3u_{n+1}+2u_n=4^n$ given $u_0=0$ and $u_1=1$ [AU A/M 2014]Refer page No.37

22. Solve the Equation $y_{n+2}-3y_{n+1}+2y_n=0$ given that $y_0=0, y_1=1$ [AU N /D 2014, A/M 2015] Refer Page No.39

23. Using Z – transforms, Solve the equation $y_{n+2}-7y_{n+1}+12y_n=2^n$ **given that** $y_0=0, y_1=0$ Refer Page No.39 **[AU M/J 2017]**

24. Form the difference equation from $y_n=A2^n+Bn$ Refer Page No: 40

25. Form the difference equation by eliminating the constants from $y_n=a2^n+b(-2)^n$ Refer Page No:41

26. Form the difference equation by eliminating the constants from $y_n=[A+Bn]2^n$ [AU N/D 2013] Refer Page No:41

27. Find the z transform of $\dfrac{1}{(n+1)(n+2)}$ [AU N/D 2013] Refer Page No:41

28. Find $z^{-1}\left[\dfrac{z(z+1)}{(z-1)^3}\right]$ [AU N/D 2007, A/M 2000, 2005] Refer Page No:42

29. Find $z^{-1}\left[\dfrac{z}{(z-1)(z^2+1)}\right]$ [AU A/M 2007, 2008 N/D 2000] Refer Page No:43

30. Find $z^{-1}\left[\dfrac{z^2}{z^2+4}\right]$ [AU N/D 2010, A/M 2003,2004] Refer Page No:44

31. Using residue method $z^{-1}\left[\dfrac{z}{(z^2-2z+2)}\right]$

Refer Page No.45 [AU M/J 2007,'08,'15,'16]

32. Using the inversion integral method (Residue Theorem), find the inverse Z-transform of $U(z)=\dfrac{z^2}{(z+2)(z^2+4)}$ Refer page No.45 [AU N/D 2015]

33. Evaluate $Z^{-1}\left[\dfrac{9z^3}{(3z-1)^2(z-2)}\right]$, using calculus of residues.

Refer Page No:46 **[AU N/D 2016]**

34. State and Prove the Final Value Theorem.

(OR)

If $z|f(t)|=F(z)$ then $\lim\limits_{t\to\infty} f(t)=\lim\limits_{z\to 1}(z-1)F(Z)$

[AU A/M 2007] Refer Page No:47

35. State and Prove the Second Shifting Theorem.

(OR)

Prove that $z|f(n+1)|=zF(Z)-Zf(0)$ [AU A/M 2013] Refer Page No:48

36. Find $Z[\cos n\theta]$ .and $Z[\sin n\theta]$.

[AU N/D 2002, A/M 2005.'08 '13 '16] Refer Page No:48

37. Find the Z-transform of $\sin^2\left(\dfrac{n\pi}{4}\right)$ and $\cos\left(\dfrac{n\pi}{2}+\dfrac{\pi}{4}\right)$ [AU N/D 2012]

Refer Page No.49

38. Find the Z-transforms of $r^n\cos n\theta$ and $e^{-at}\cos bt$ Refer page No.50 [AU A/M 2014]

39. Find Z – Transform of $\dfrac{2n+3}{(n+1)(n+2)}$ Refer Page No:51 **[AU M/J 2017]**

40. Find $Z^{-1}\left[\dfrac{4z^3}{(2z-1)^2(z-1)}\right]$ by the method of partial fractions.

Refer Page No:52 **[AU M/J 2017]**

41. Find $z^{-1}\left[\dfrac{(z^2+z)}{(z-1)(z^2+1)}\right]$ Refer page No.52 [AU N/D 2014]

42. If $U(z)=\dfrac{z^3+z}{(z-1)^3}$, find the value of u_0, u_1 and u_2. Refer page No.53

[AU N/D 2015]

43.Find the inverse Z- transform of $\frac{z^2-3z}{(z-5)(z+2)}$ using residue theorem.

[AU A/M 2018] [P.NO: 54]

44. Find the inverse Z –transform of $\frac{z^3}{(z-1)^2(z-2)}$ by method of partial fractions.

[AU N/D 2017] [P.NO: 54]

Important Questions (Z -Transforms)
PART -A

1. Define Z- transform [May 2009, Apr 2007]
2. Prove that $z|a^n| = \frac{z}{z-a}$ $|z| > |a|$ [Apr-2009, 2005, June-2000, Apr-99]
3. Find z[n] [Dec 2010, Apr 2007, Apr 2000, MAY 2013]
4. Find Z{f(n)} if f(n) $= \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$ [Dec 2008, May 2007]
5. Find $Z\left[\frac{a^n}{n!}\right]$ in Z- transform [Nov -2005]
6. Find $Z[a^{|n|}]$ [Dec-2010, Apr-2009, May-2002,Apr -2000]
7. Find $Z(-1)^n$ [May 2010 , Mar 2005]
8. Find $Z\left[e^{-an}\right]$ [Apr. 2008 and 2007]
9. Find $Z[a^{n-1}]$
10. Find $Z\left[\frac{1}{n(n+1)}\right]$
11. Find $Z\left[sin\frac{n\pi}{2}\right]$ [Dec 2009 ,Apr 2007]
12. Prove that Z [n f (n)] $= -Z\frac{d}{dz}$ $\{F(Z)\}$
13. Find the Z transform of ${}^nc_k$
14. Find $Z(n^2)$ [A.U. MAY/JUNE 2014]
15. Find $Z[an^2+bn+c]$
16. Find $Z[n(n-1)]$ [May 2009, Apr2007]
17. Find $Z[a^n \sin n\theta]$ [Apr 2005, 2007]
18. Find $Z(na^n)$ [Nov/Dec-2010]
19. Find $z[a^n/n!]$
20. Find the Z-Transform of (1/n!)

21 . Find the Z-Transform of (n+1)(n+2)

22. Find the Z-Transform of 1/n [A.U. Nov 2013]

23. Define Unit step sequence

24. Define unit impulse sequence [May 2005, Apr 2004]

25. Prove Initial value theorem [Dec2007, Apr2004]

26. State and Prove first shifting theorem

27. State and Prove second shifting theorem

28. *If* $F(Z)=\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$ Find f (0)

29. *.Find the final value of the function* $F(z)=\dfrac{1+z^{-1}}{1-0.25z^{-2}}$ [May2009, Apr2007]

30. Define inverse z- transform with example

31. State initial and final value theorem in Z-transform

32. Define convolution of two sequences. [Apr 2005&2003]

33. State convolution theorem on Z-transforms [A.U. MAY/JUNE 2014]

34. Form a difference equation by eliminating the arbitrary constant A form $Y_n = A.3^n$ [May 2009, Apr 2007, Nov 2010]

35. Form a difference equation by eliminating arbitrary constant from $U_n = a\,2^{n+1}$

36. Find the inverse Z-Transform of $\left[\dfrac{z}{(z+1)^2}\right]$ [A.U. NOV 2013]

37. Find the inverse Z-Transform of $\left[\dfrac{z}{(z+1)(z+2)}\right]$ [A.U. May 2013]

38. Find the inverse Z-Transform of $\left[\dfrac{z}{(z+1)(z+2)}\right]$

39. *Find* $z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$ [Nov 2008, Dec 2007]

40. If $f(z)=\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$ find f(0)

PART – B

1. a)Using Convolution Theorem, Find the $Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right]$

[AU N/D 2010,2015] Refer Page No: 18

b) Using Convolution theorem, find the value of $Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right]$

[AU N/D 2010] Refer Page No: 18

2. a) Using Convolution theorem finds the inverse z- transform $\left[\frac{14z^2}{(7z-1)(2z-1)}\right]$

Refer Page No:19

b) Using Convolution Theorem Find the value of $Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\right]$

[AU A/M 2012] Refer Page No:20

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3. a) Using Convolution theorem, Find $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$ [AU N/D 2007, M/J 2016]

(or) Refer Page No: 21

Find $Z^{-1}\left[(1-az^{-1})^{-2}\right]$

b) Using convolution theorem find the inverse Z- transform $\left[\frac{z^2}{(z-1)(z-3)}\right]$

[AU N/D 2006, 2013] Refer Page No: 22

4.a)Find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$ Using Convolution Theorem. [AU A/M 2010]

Refer Page No:22

b) Using Convolution Theorem, Find the $z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Refer Page No:23 [AU A/M 2013, 2014]

5. a)Solve $y_{n+2}+4y_{n+1}+3y_n=2^n$ with $y_0=0$ and $y_1=1$ using Z-transform

[AU N/D 2010, A/M 2005,2006] Refer Page No:24

b) Solve the Equation $y_{n+2}-3y_{n+1}+2y_n=2^n$ given that $y_0=y_1=0$

[AU A/M 2008, 2009, 2011] Refer Page No:25

6. a)Using Z-transform solve $U(n+2)-5U(n+1)+6U(n)=4^n$ given that

$U(0)=0, U(1)=1$ [AU A/M 2005 N/D 2009] Refer Page No:26

b) Solve by using Z –transform $y_{n+2}+4y_{n+1}+3y_n=3^n$, given that $y_0=0$;

$y_1 = 1$. [AU A/M 2009, 2010 N/D 2015] Refer Page No:27

7.a)Solve the differential equation $y(k+2)-4y(k+1)+4y(k)=0$ where $\mathbf{y(0)=1,\ y(1)=0}$ [AU N/D 2005, 2009, A/M 2009] Refer Page No:28

b) Solve the difference equation $y(n)+3y(n-1)-4y(n-2)=0$, $n \geq 2$, given that $\mathbf{y(0)=3}$ **and** $\mathbf{y(1)=2}$. [AU A/M 2006] Refer Page No:29

8. a) Solve using Z-transform, $y_{n+2}-3y_{n+1}-10y_n=0$ given $y_0=1$ and $y_1=0$ [AU N/D 2002,'08,'09, A/M 2013, 2014] Refer Page No:30

b) Solve the difference equation $y(k+2)+y(k)=1$, $y(0)=y(1)=0$, using Z-transform [AU A/M 2012] Refer Page No:31

9. a) Solve the difference equation $y_{n+2}+y_n=2$, $y_0=y_1=0$, using Z-transform. [AU M/J 2016] Refer Page No:32

b) Solve $y_{n+2}+y_n=n.2^n$,using Z-transform [AU A/M 2012] Refer Page No:33

10.a) Solve the difference using Z-transform $y_{(n+3)}-3y_{(n+1)}+2y_{(n)}=0$ given that $y_0=0, y_1=0, y_2=8$ [AU N/D 2012] Refer Page No:35

b) Solve $y_{(n+2)}+6y_{(n+1)}+9y_{(n)}=2^n$ given that $y_0=y_1=0$ [AU N/D 2012] Refer Page No:36

11. a) Solve using Z-transform, $u_{n+2}-3u_{n+1}+2u_n=4^n$ given $u_0=0$ and $u_1=1$ [AU A/M 2014]Refer page No.37

b) Solve the Equation $y_{n+2}-3y_{n+1}+2y_n=0$ given that $y_0=0, y_1=1$ [AU N /D 2014, A/M 2015] Refer Page No.39

12.a) Form the difference equation from $y_n=A2^n+Bn$ Refer Page No: 40

b) Form the difference equation by eliminating the constants from $y_n=a2^n+b(-2)^n$ Refer Page No:41

13. a) Form the difference equation by eliminating the constants from $y_n=[A+Bn]2^n$ [AU N/D 2013] Refer Page No:41

b) Find the z transform of $\dfrac{1}{(n+1)(n+2)}$ [AU N/D 2013] Refer Page No:41

14. a) Find $z^{-1}\left[\dfrac{z(z+1)}{(z-1)^3}\right]$ [AU N/D 2007, A/M 2000, 2005] Refer Page No:42

b) Find $z^{-1}\left[\dfrac{z}{(z-1)(z^2+1)}\right]$ [AU A/M 2007, 2008 N/D 2000] Refer Page No:43

15.a) Find $z^{-1}\left[\dfrac{z^2}{z^2+4}\right]$ [AU N/D 2010, A/M 2003,2004] Refer Page No:44

b) Using residue method $z^{-1}\left[\dfrac{z}{(z^2-2z+2)}\right]$ Refer Page No.45 [AU M/J 2007,'08,'15,'16]

16.a) Using the inversion integral method (Residue Theorem), find the inverse Z-transform of $U(z)=\dfrac{z^2}{(z+2)(z^2+4)}$ Refer page No.45 [AU N/D 2015]

b) State and Prove the Final Value Theorem.

(OR)

If $z\left|f(t)\right| = F(z)$ then $\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)F(Z)$

[AU A/M 2007] Refer Page No:47

17. a) State and Prove the Second Shifting Theorem.

(OR)

$\Pr ove\ \ that\ \ z\left|f(n+1)\right| = zF(Z) - Zf(0)$ [AU A/M 2013] Refer Page No:48

b) Find $Z[\cos n\theta]$ .and $Z[\sin n\theta]$.

[AU N/D 2002, A/M 2005.'08 '13 '16] Refer Page No:48

18. a) Find the Z-transform of $\sin^2\left(\frac{n\pi}{4}\right)$ and $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ [AU N/D 2012]Refer Page No.49

b) Find the Z-transforms of $r^n \cos n\theta$ and $e^{-at}\cos bt$ Refer page No.50 [AU A/M 2014]

19. a) Find $z^{-1}\left[\frac{(z^2+z)}{(z-1)(z^2+1)}\right]$ Refer page No.52 [AU N/D 2014]

b) If $U(z) = \frac{z^3+z}{(z-1)^3}$, find the value of u_0, u_1 *and* u_2.Refer page No.53

[AU N/D 2015]

Reg. No. :

Question Paper Code : 51571

B.E./B.Tech. DEGREE EXAMINATION, MAY/JUNE 2014.

Third Semester

Civil Engineering

MA 2211/MA 31/MA 1201 A/CK 201/080100008/080210001/10177 MA 301 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS/ MATHEMATICS — III

(Common to all branches)

(Regulation 2008/2010)

Time : Three hours | Maximum : 100 marks

Answer ALL questions.

PART A — ($10 \times 2 = 20$ marks)

1. State the conditions for a function $f(x)$ to be expanded as a Fourier series in a given interval.

2. Expand $f(x) = 1$ as a half range sine series in the interval $(0, \pi)$.

3. Find the Fourier sine transform of $f(x) = \frac{1}{x}$.

4. State the Fourier integral theorem.

5. Form the PDE by eliminating the arbitrary constants a, b from the relation $z = ax^3 + by^3$.

6. Solve : $(D^4 - D'^4)z = 0$.

7. Write all the solutions of the one-dimensional wave equation $y_{tt} = a^2 y_{xx}$.

8. State the assumptions in deriving the one-dimensions heat flow equation (unsteady state).

9. Find the Z-transform of n^2.

10. State the convolution theorem on Z-transforms.

PART B — ($5 \times 16 = 80$ marks)

11. (a) (i) Expand $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \dfrac{2x}{\pi}, & 0 < x < \pi \end{cases}$ as a full range Fourier series in the interval $(-\pi, \pi)$. Hence deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\infty = \dfrac{\pi^2}{8}$. (8)

(ii) Find the half-range sine series of $f(x) = 4x - x^2$ in the interval $(0, 4)$. Hence deduce the value of the series $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \ldots\infty$. (8)

Or

(b) (i) Expand $f(x) = \sin x$ as a complex form Fourier series in $(-\pi, \pi)$. (8)

(ii) Compute the first three harmonics of the Fourier series for $f(x)$ from the following data : (8)

x :	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
$f(x)$:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

12. (a) (i) Find the Fourier transform of $e^{-a|x|}$, $a > 0$ and hence deduce that

(1) $\displaystyle\int_0^\infty \frac{\cos xt}{a^2 + t^2}\, dt = \frac{\pi}{2a} e^{-a|x|}$

(2) $F\left\{xe^{-a|x|}\right\} = i\sqrt{\dfrac{2}{\pi}}\, \dfrac{2as}{\left(s^2 + a^2\right)^2}$, here F stands for Fourier transform. (8)

(ii) Solve for $f(x)$ from the integral equation (8)

$$\int_0^\infty f(x) \sin sx\, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2. \end{cases}$$

Or

(b) (i) Find the Fourier transform of $f(x) = \frac{1}{\sqrt{|x|}}$. (8)

(ii) Using Parseval's identity evaluate the following integrals

(1) $\int_0^\infty \frac{dx}{(a^2 + x^2)^2}$

(2) $\int_0^\infty \frac{x^2}{(a^2 + x^2)^2} dx$ where $a > 0$. (8)

13. (a) (i) Form the PDE by eliminating the arbitrary function from the relation $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. (8)

(ii) Solve the Lagrange's equation $(x + 2z)p + (2xz - y)q = x^2 + y$. (8)

Or

(b) (i) Solve : $x^2p^2 + y^2q^2 = z^2$. (8)

(ii) Solve : $(D^2 + DD' - 6D'^2)z = y \cos x$. (8)

14. (a) A string is stretched and fastened to points at a distance 'l' apart. Motion is started by displacing the string in the form $y = a \sin\left(\frac{\pi x}{l}\right)$, $0 < x < l$, from which it is released at time $t = 0$. Find the displacement at any time t. (16)

Or

(b) An infinitely long rectangular plate with insulated surfaces is 10 cm wide. The two long edges and one short edge are kept at 0°C, while the other short edge $x = 0$ is kept at temperature

$u = 20y, \quad 0 \le y \le 5$

$u = 20(10 - y), \quad 5 < y \le 10$.

Find the steady state temperature distribution in the plate. (16)

15. (a) (i) Find the Z-transforms of $r^n \cos n\theta$ and $e^{-at} \cos bt$. (8)

(ii) Solve $u_{n+2} - 3u_{n+1} + 2u_n = 4^n$, given that $u_0 = 0$, $u_1 = 1$. (8)

Or

(b) (i) Using convolution theorem find inverse Z-transform of $\frac{z^2}{(z-a)(z-b)}$. (8)

(ii) Solve $y_{n+2} - 3y_{n+1} - 10y_n = 0$, given $y_0 = 1$, $y_1 = 0$. (8)

Reg. No. : 4 2 1 6 1 3 1 1 4 0 5 0

Question Paper Code : 97107

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2014.

Third Semester

Civil Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulation 2013)

Time : Three hours | Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Form the partial differential equation by eliminating the arbitrary function f from $z = f\left(\frac{y}{x}\right)$. Pg. No. 6, Q. NO. 16

2. Find the complete solution of $p+q=1$.

3. State the sufficient conditions for existence of Fourier series. Pg. No. 1 Q. NO. 1

4. If $(\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ in $0 < x < 2\pi$, then deduce that value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Pg. No. 2, Q

5. Classify the partial differential equation $(1-x^2)z_{xx} - 2xy\,z_{xy} + (1-y^2)z_{yy} + xz_x + 3x^2 y\,z_y - 2z = 0$. P.g No: 10 Q. No 44 (SM

6. Write down the various possible solutions of one dimensional heat flow equation. Pg. NO. 7, Q. NO. 28

7. State and prove modulation theorem on Fourier transforms. Pg. No. 5, Q. NO. 13

8. If $F\{f(x)\} = F(s)$, then find $F\{e^{iax} f(x)\}$. Pg. No. 11, Q. No. 27

9. Find the Z transform of n. Pg. NO. 2, Q. NO. 3

10. State initial value theorem on Z transforms. Pg. No. 16, Q. NO. 44

PART B — (5 × 16 = 80 marks)

11. (a) (i) Find the singular solution of $z = px + qy + p^2 - q^2$. Pg. No. 39, Q. NO. 22 (SM (8)

(ii) Solve $(D^2 - 2DD')z = x^3 y + e^{2x-y}$. Pg. NO. 37, Q. NO. 20 (SM) (8)

Or

(b) (i) Solve $x(y-z)p + y(z-x)q = z(x-y)$. Pg. No. 21, Q. NO. 1 (8)

(ii) Solve $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y)$. Pg. No. 44, Q. No. 28 (SM) (8)

5. K. KALAIYARASAN, (signature)

12. (a) (i) Find the Fourier series of $f(x) = x^2$ in $-\pi < x < \pi$. Hence deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Pg. No. 29, Q. No. 8 (8)

(ii) Find the half range cosine series expansion of $(x-1)^2$ in $0<x<1$. (8) Pg. No. 38 Q. No. 16

Or

(b) (i) Compute the first two harmonics of the Fourier series of $f(x)$ from the table given. (8)

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Pg. No. 46 Q. No. 26

(ii) Obtain the Fourier cosine series expansion of $f(x) = x$ in $0 < x < 4$. Hence deduce the value of $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \ldots$ to ∞. Pg. No. 35 Q. No. 13 (8) (SM)

13. (a) If a tightly stretched string of length l is initially at rest in equilibrium position and each point of it is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$, $0 < x < l$, determine the transverse displacement $y(x,t)$. (16) Pg. No. 31 Q. No. 9

Or

(b) A square plate is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. Its surfaces are insulated and the temperature along $y = b$ is kept at 100°C. while the temperature along other three edges are at 0°C. Find the steady - state temperature at any point in the plate. (16)

14. (a) Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & \text{if } |x|<1 \\ 0, & \text{otherwise} \end{cases}$. Hence, deduce the values (i) $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt$ (ii) $\int_0^{\infty} \frac{\sin^4 t}{t^4} dt$. Pg. No. 21, Q. No. 2 (16)

Or

(b) (i) Find the Fourier transform of $e^{-a^2x^2}$, $a > 0$. Hence show that $e^{\frac{-x^2}{2}}$ is self reciprocal under the Fourier transform. Pg. No. 30, Q. No. 7 (8)

(ii) Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, using Fourier transforms. Pg. No. 44 Q. No. 21 (8)

15. (a) (i) Find $Z(\cos n\theta)$ and $Z(\sin n\theta)$. Pg. No. 33, Q. No. 21 (8)

(ii) Using Z-transforms, solve $u_{n+2} - 3u_{n+1} + 2u_n = 0$ given that $u_0 = 0, u_1 = 1$. Pg. No. 34, Q. No. 22 (SM) (8)

Or

(b) (i) Find the Z-transform of $\frac{1}{n(n+1)}$, for $n \geq 1$. Pg No. 5, Q No. 11 (8)

(ii) Find the inverse Z-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$, using partial fraction. (8)

Pg. No. 30 Q. No. 17 (SM)

Reg. No. | | | | | | | | | | | | |

Question Paper Code : 77191

B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2015.

Third Semester

Civil Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulation 2013)

Time : Three hours　　　　　　　　　　Maximum : 100 marks

Answer ALL questions.

PART A — ($10 \times 2 = 20$ marks)

1. Form the partial differential equation by eliminating the arbitrary constants a and b from $\log(az-1) = x + ay + b$

2. Find the complete solution of $q = 2px$.

3. The instantaneous current 'i' at time t of an alternating current wave is given by $i = I_1 \sin(\omega t + \alpha_1) + I_3 \sin(3\omega t + \alpha_3) + I_5 \sin(5\omega t + \alpha_5) + \cdots$. Find the effective value of the current 'i'.

4. If the Fourier series of the function $f(x) = x, -\pi < x < \pi$ with period 2π is given by $f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right)$, then find the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

5. Classify the partial differential equaiton $(1-x^2)z_{xx} - 2xyz_{xy} + (1-y^2)z_{yy} + xz_x + 3x^2yz_y - 2z = 0$.

6. A rod 30 cm long has its ends A and B kept at 20°C and 80°C respectively until steady state conditions prevail. Find this steady state temperature in the rod.

7. If the Fourier transform of $f(x)$ is $\Im(f(x)) = F(s)$, then show that $\Im(f(x-a)) = e^{ias}F(s)$.

8. Find the Fourier sine transform of $1/x$.

9. If $Z(x(n)) = X(z)$, then show that $Z(a^n x(n)) = X\left(\frac{z}{a}\right)$.

10. State the convolution theorem of Z-transforms.

PART B — $(5 \times 16 = 80$ marks)

11. (a) (i) Solve: $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$. (8)

(ii) Solve: $(D^2 - 3DD' + 2D'^2)z = (2+4x)e^{x+2y}$. (8)

Or

(b) (i) Obtain the complete solution of $p^2 + x^2y^2q^2 = x^2z^2$. (8)

(ii) Solve $z = px + qy + p^2q^2$ and obtain its singular solution. (8)

12. (a) (i) Find the half-range sine series of $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$. Hence deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$. (10)

(ii) Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$. (6)

Or

(b) (i) Find the Fourier series of $f(x) = |\sin x|$ in $-\pi < x < \pi$ of periodicity 2π. (8)

(ii) Compute upto the first three harmonics of the Fourier series of $f(x)$ given by the following table: (8)

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

13. (a) Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ subject to the conditions: $u(0,t) = 0 = u(l,t), t \geq 0$;

$u(x,0) = \begin{cases} x, & 0 \leq x \leq l/2 \\ l - x, & l/2 \leq x \leq l \end{cases}$. (16)

Or

(b) A string is stretched and fastened to two points that are distant l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time t=0. Find the displacement at any point of the string at a distance x from one end at any time t. (16)

14. (a) Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$. Using Parseval's identity, prove that $\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$. (16)

Or

(b) (i) Show that the function $e^{-x^2/2}$ is self-reciprocal under Fourier transform by finding the Fourier transform of $e^{-a^2 x^2}, a > 0$. (10)

(ii) Find the Fourier cosine transform of x^{n-1}. (6)

15. (a) (i) Find $Z(r^n \cos n\theta)$ and $Z^{-1}\left[\left(1 - az^{-1}\right)^{-2}\right]$. (8)

(ii) Using convolution theorem, find $Z^{-1}\left[\frac{z^2}{(z - 1/2)(z - 1/4)}\right]$. (8)

Or

(b) (i) Using Z – transform, solve the difference equation $x(n+2) - 3x(n+1) + 2x(n) = 0$ given that $x(0) = 0, x(1) = 1$. (8)

(ii) Using residue method, find $Z^{-1}\left[\frac{z}{z^2 - 2z + 2}\right]$. (8)

Reg. No. : | | | | | | | | | | | | |

Question Paper Code : 27327

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2015.

Third Semester

Civil Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulation 2013)

Time : Three hours

Maximum : 100 marks

Answer ALL questions.

PART A — ($10 \times 2 = 20$ marks)

1. Construct the partial differential equation of all spheres whose centres lie on the Z – axis, by the elimination of arbitrary constants.

2. Solve $(D+D'-1)(D-2D'+3)z=0$.

3. Find the root mean square value of $f(x)=x(l-x)$ in $0 \le x \le l$.

4. Find the sine series of function $f(x)=1$, $0 \le x \le \pi$.

5. Solve $3x\frac{\partial u}{\partial x}-2y\frac{\partial u}{\partial y}=0$; by method of separation of variables.

6. Write all possible solutions of two dimensional heat equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

7. If $F(s)$ is the Fourier Transform of $f(x)$, prove that $F\{f(ax)\}=\frac{1}{a}F\left(\frac{s}{a}\right), a \ne 0$.

8. Evaluate $\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds$ using Fourier Transforms.

9. Find the Z – transform of $\frac{1}{n+1}$.

10. State the final value theorem. In Z transform.

PART B — (5 × 16 = 80 marks)

11. (a) (i) Find complete solution of $z^2(p^2+q^2)=(x^2+y^2)$. (8)

(ii) Find the general solution of $(D^2+2DD'+D'^2)z=2\cos y - x\sin y$. (8)

Or

(b) (i) Find the general solution of $(z^2-y^2-2yz)p+(xy+zx)q=(xy-zx)$. (8)

(ii) Find the general solution of $(D^2+D'^2)z=x^2y^2$. (8)

12. (a) (i) Find the Fourier series expansion the following periodic function of period 4 $f(x)=\begin{cases}2+x & -2\le x\le 0\\ 2-x & 0<x\le 2\end{cases}$. Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....=\frac{\pi^2}{8}$. (8)

(ii) Find the complex form of Fourier series of $f(x)=e^{ax}$ in the interval $(-\pi,\pi)$ where a is a real constant. Hence, deduce that $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}=\frac{\pi}{a\sinh a\pi}$. (8)

Or

(b) (i) Find the half range cosine series of $f(x)=(\pi-x)^2, 0<x<\pi$. Hence find the sum of series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+....$ (8)

(ii) Determine the first two harmonics of Fourier series for the following data. (8)

x :	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
$f(x)$:	1.98	1.30	1.05	1.30	–0.88	–0.25

13. (a) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $v=\begin{cases}\frac{2kx}{l} & in\ \ 0<x<\frac{l}{2}\\ \frac{2k(l-x)}{l} & in\ \ \frac{1}{2}<x<l\end{cases}$. Find the displacement of the string at any distance x from one end at any time t. (16)

Or

(b) A bar 10 cm long with insulated sides has its ends A and B maintained at temperature $50°C$ and $100°C$, respectively, until steady state conditions prevails. The temperature at A is suddenly raised to $90°C$ and at the same time lowered to $60°C$ at B. Find the temperature distributed in the bar at time t. (16)

14. (a) (i) Find the Fourier sine integral representation of the function $f(x)=e^{-x}\sin x$. (8)

(ii) Find the Fourier cosine transform of the function $f(x)=\dfrac{e^{-ax}-e^{-bx}}{x}, x>0$. (8)

Or

(b) (i) Find the Fourier transform of the function $f(x)=\begin{cases}1-|x|, & |x|\le 1\\ 0, & |x|>1\end{cases}$.

Hence deduce that $\displaystyle\int_0^\infty\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$. (8)

(ii) Verify the convolution theorem for Fourier transform if $f(x)=g(x)=e^{-x^2}$. (8)

15. (a) (i) If $U(z)=\dfrac{z^3+z}{(z-1)^3}$, find the value of u_0, u_1 and u_2. (8)

(ii) Use convolution theorem to evaluate $z^{-1}\left\{\dfrac{z^2}{(z-3)(z-4)}\right\}$. (8)

Or

(b) (i) Using the inversion integral method (Residue Theorem), find the inverse Z- transform of $U(z)=\dfrac{z^2}{(z+2)(z^2+4)}$. (8)

(ii) Using the Z- transform solve the difference equation $u_{n+2}+4u_{n+1}+3u_n=3^n$ with $u_0=0, u_1=1$. (8)

Reg. No. [ ]

Question Paper Code : 57502

B.E./B. Tech. DEGREE EXAMINATION, MAY/JUNE 2016

Third Semester

Civil Engineering

MA 6351 – TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulations 2013)

Time : Three Hours **Maximum : 100 Marks**

Answer ALL questions.

PART – A (10 × 2 = 20 Marks)

1. Form the partial differential equation by eliminating the arbitrary functions from $f(x^2 + y^2, z - xy) = 0$.

2. Find the complete solution of the partial differential equation $p^3 - q^3 = 0$.

3. Find the value of the Fourier series of $f(x) = \begin{cases} 0 & \text{in } (-c, 0) \\ 1 & \text{in } (0, c) \end{cases}$ at the point of discontinuity $x = 0$.

4. Find the value of b_n in the Fourier series expansion of $f(x) = \begin{cases} x + \pi & \text{in } (-\pi, 0) \\ -x + \pi & \text{in } (0, \pi) \end{cases}$

5. Classify the partial differential equation $u_{xx} + u_{xy} = f(x, y)$.

6. Write down all the possible solutions of one dimensional heat equation.

7. State Fourier integral theorem.

8. Find the Fourier transform of a derivative of the function $f(x)$ if $f(x) \to 0$ as $x \to \pm\infty$.

9. Find $Z\left\{\frac{1}{n!}\right\}$

10. Find $Z\{(\cos\theta + i\sin\theta)^n\}$.

PART – B (5 × 16 = 80 Marks)

11. (a) (i) Solve the equation $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. **(8)**

(ii) Find the singular integral of the equation $z = px + qy + \sqrt{1 + p^2 + q^2}$. **(8)**

OR

(b) (i) Solve : $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$. **(8)**

(ii) Solve : $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$ **(8)**

12. (a) (i) Find the Fourier series of $f(x) = x$ in $-\pi < x < \pi$. **(6)**

(ii) Find the Fourier series expansion of $f(x) = |\cos x|$ in $-\pi < x < \pi$. **(10)**

OR

(b) (i) Find the half range sine series of $f(x) = x\cos\pi x$ in $(0, 1)$. **(8)**

(ii) Find the Fourier cosine series up to third harmonic to represent the function given by the following data : **(8)**

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

13. (a) Find the displacement of a string stretched between two fixed points at a distance of $2l$ apart when the string is initially at rest in equilibrium position and points of the string are given initial velocities v where $v = \begin{cases} \dfrac{x}{l} & \text{in } (0, l) \\ \dfrac{2l - x}{l} & \text{in } (l, 2l) \end{cases}$, x being the distance measured from one end. **(16)**

OR

(b) A long rectangular plate with insulated surface is l cm wide. If the temperature along one short edge is $u(x, 0) = k(lx - x^2)$ for $0 < x < l$, while the other two long edges $x = 0$ and $x = 1$ as well as the other short edge are kept at 0 °C, find the steady state temperature function $u(x, y)$. **(16)**

14. (a) Find the Fourier cosine and sine transform of $f(x) = e^{-ax}$ for $x \geq 0$, $a > 0$. Hence deduce the integrals $\int_0^\infty \frac{\cos sx}{a^2 + s^2} ds$ and $\int_0^\infty \frac{s \sin sx}{a^2 + s^2} ds$. **(16)**

OR

(b) (i) Find the Fourier transform of $f(x) = e^{-\frac{x^2}{2}}$ in $(-\infty, \infty)$. **(8)**

(ii) Find the Fourier transform of $f(x) = 1 - |x|$ if $|x| < 1$ and hence find the value of $\int_0^\infty \frac{\sin^4 t}{t^4} dt$. **(8)**

15. (a) (i) Find the Z-transforms of $\cos \frac{n\pi}{2}$ and $\frac{1}{n(n+1)}$. **(8)**

(ii) Using convolution theorem, evaluation $Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\}$. **(8)**

OR

(b) (i) Find the inverse Z-transform of $\frac{z}{z^2-2z+2}$ by residue method. **(8)**

(ii) Solve the difference equation $y_{n+2} + y_n = 2$, given that $y_0 = 0$ and $y_1 = 0$ by using Z-transforms. **(8)**

Reg. No. | | | | | | | | | | | |

Question Paper Code : 80608

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2016

Third Semester

Civil Engineering

MA 6351 – TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to all branches except Environmental Engineering, Textile Chemistry, Textile Technology, Fashion Technology and Pharmaceutical Technology)

(Regulations 2013)

Time : Three hours — Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Find the PDE of all spheres whose centers lie on the x-axis.

2. Find the complete integral of $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$.

3. State the Dirichlet's conditions for a function $f(x)$ to be expanded as a Fourier series.

4. Expand $f(x) = 1$, in $(0, \pi)$ as a half-range sine series.

5. State the assumptions in deriving one-dimensional wave equation.

6. State the three possible solutions of the one-dimensional heat flow (unsteady state) equation.

7. State change of scale property on Fourier transforms.

8. Find the infinite Fourier sine transform of $f(x) = \frac{1}{x}$

9. State convolution theorem on Z-transform.

10. Find $Z\left[\frac{1}{n(n+1)}\right]$.

PART B — (5 × 16 = 80 marks)

11. (a) (i) Find the partial differential equations of all planes which are at a constant distance 'k' units from the origin. (8)

(ii) Solve the Lagrange's equation $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$. (8)

Or

(b) (i) Form the PDE by eliminating the arbitrary functions 'f' and 'φ' from the relation $z = xf\left(\frac{y}{x}\right) + y\varphi(x)$. (8)

(ii) Solve $(D^2 + DD' - 6D'^2)z = y\cos x$. (8)

12. (a) (i) Expand $f(x) = x^2$ as a Fourier series in the interval $(-\pi, \pi)$ and hence deduce that $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$. (8)

(ii) Obtain the constant term and the coefficient of the first sine and cosine terms in the Fourier expansion of y as given in the following table: (8)

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Or

(b) (i) Expand $f(x) = e^{-ax}, -\pi < x < \pi$ as a complex form Fourier series. (8)

(ii) Expand $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \end{cases}$ as a series of cosines in the interval $(0,2)$. (8)

13. (a) A tightly stretched string of length 'l' with fixed end points is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity $y_t(x,0) = v_0 \sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)$, where $0 < x < l$. Find the displacement of the string at a point, at a distance x from one end at any instant 't'. (16)

Or

(b) A square plate is bounded by the lines $x=0, x=20, y=0, y=20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,20) = x(20-x), 0 < x < 20$, while the other three edges are kept at 0°C. Find the steady state temperature distribution $u(x,y)$ in the plate. (16)

14. (a) (i) Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ and hence deduce that $\int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt = \frac{\pi}{3}$. (8)

(ii) Find the infinite Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ hence deduce the infinite Fourier sine transform of $\frac{1}{x}$. (8)

Or

(b) (i) Find the infinite Fourier transform of $e^{-a^2x^2}$ hence deduce the infinite Fourier transform of $e^{-x^2/2}$. (8)

(ii) Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\, dx = e^{-\lambda}$, where $\lambda > 0$. (8)

15. (a) (i) Find

(1) $Z[n^3]$ (2) $Z[e^{-t}t^2]$. (4+4)

(ii) Evaluate $Z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right]$, using calculus of residues. (8)

Or

(b) (i) Using convolution theorem, evaluate $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$. (8)

(ii) Using Z-transform, solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given that $y_0 = y_1 = 0$. (8)

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Reg. No. : | | | | | | | | | | | | |

Question Paper Code : 72068

B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2017.

Third Semester

Mechanical Engineering

MA 6351 — TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to Mechanical Engineering (Sandwich)/Aeronautical Engineering/ Agriculture Engineering/Automobile Engineering/B.E. Biomedical Engineering/ B.E. Civil engineering/B.E. Computer Science and Engineering/Electrical and Electronics Engineering/Electronics and Communication Engineering/Electronics and Instrumentation Engineering/Geoinformatics Engineering/Industrial Engineering/Industrial Engineering and Management/Instrumentation and Control Engineering/Manufacturing Engineering/Marine Engineering/Materials Science and Engineering/Mechanical and Automation Engineering/Mechatronics Engineering/Medical Electronics Engineering/Petrochemical Engineering/ Production Engineering/Robotics and Automation Engineering/ Biotechnology/ Chemical Engineering/Chemical and Electrochemical Engineering/Food Technology/Information Technology/Petrochemical Technology/Petroleum Engineering/Plastic Technology/Polymer Technology)

(Regulations 2013)

Time : Three hours | Maximum : 100 marks

Answer ALL questions.

PART A — (10 × 2 = 20 marks)

1. Form the partial differential equation by eliminating arbitrary function 'f' from $z = e^{ay} f(x+by)$.

2. Solve $(D^3 - D^2D' - 8DD'^2 + 12D'^3)z = 0$.

3. State the sufficient condition for a function $f(x)$ to be expressed as a Fourier series.

4. If the Fourier series of the function $f(x) = x + x^2$, in the interval $(-\pi, \pi)$ is $\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]$, then find the value of the infinite series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ ([illegible]) (Q.81) (Singaravelu)

5. Write all possible solutions of one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.

6. Using the method of separation of variables, solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x, 0) = 6e^{-3x}$.

7. If $F(s)$ is the Fourier transform of $f(x)$, prove that $F\{f(x-a)\} = e^{ias}F(s)$.

8. Find Fourier Sine transform of $\frac{1}{x}$.

9. Find the Z-transform of a^n.

10. State initial and final value theorems on Z-transforms.

PART B — (5 × 16 = 80 marks)

11. (a) (i) Find the general solution of $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. (8) (59) (45)

(ii) Find the general solution of $(D^2 + 2DD' + D'^2)z = x^2 y + e^{x-y}$. (8)

Or

(b) (i) Find the general solution of $z = px + qy + p^2 + pq + q^2$. (8)

(ii) Find the general solution of $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = \sin(2x + y)$. (8)

12. (a) (i) Find the Fourier series of period 2π for the function $f(x) = x\cos x$ in $0 < x < 2\pi$. (2.19) Chandrasekaran (8)

(ii) Find the Fourier series expansion for $y = f(x)$ up to second harmonic from the following data: (8)

x:	0	1	2	3	4	5
y:	9	18	24	28	26	20

Or

(b) (i) Find the Fourier half-range cosine series of $f(x) = \begin{cases} x, & \text{in } 0 < x < 1 \\ 2 - x, & \text{in } 1 < x < 2 \end{cases}$. (8)

(ii) Find the complex form of the Fourier series of $f(x) = e^{-ax}$ in, $-l < x < l$. (8)

13. (a) A tightly stretched string of length $2l$ is fastened at $x=0$ and $x=2l$. The midpoint of the string is then taken to height 'b' transversely and then released from rest in that position. Find the lateral displacement of the string. (16)

Or

(b) A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature while the other short edge $x=0$ is given by $u=\begin{cases}10y & \text{for } 0\le y\le 10\\ 10(20-y) & \text{for } 10\le y\le 20\end{cases}$ and the two long edges as well as the other short edge are kept at 0°c, find the steady state temperature distribution $u(x,y)$ in the plate. (16)

14. (a) Find the Fourier transform of $f(x)$ given by $f(x)=\begin{cases}1 & \text{for } |x|<2\\ 0 & \text{for } |x|>2\end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$ and $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$. (16)

Or

(b) (i) Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a>0$. (8)

(ii) Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ using Fourier transforms. (8)

15. (a) (i) Find Z-transform of $\frac{2n+3}{(n+1)\,(n+2)}$. (8)

(ii) Using Convolution theorem, find $Z^{-1}\left[\frac{8z^2}{(2z-1)\,(4z+1)}\right]$. (8)

Or

(b) (i) Find $Z^{-1}\left[\frac{4z^3}{(2z-1)^2\,(z-1)}\right]$, by the method of partial fractions. (8)

(ii) Using Z-transforms, solve the equation $y_{n+2}-7y_{n+1}+12y_n=2^n$, given that $y_0=y_1=0$. (8)

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Reg. No. :

Question Paper Code : 50779

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2017

Third Semester

Civil Engineering

MA 6351 – TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to Mechanical Engineering (Sandwich)/ Aeronautical Engineering/ Agriculture Engineering/ Automobile Engineering/ Biomedical Engineering/ Computer Science and Engineering/ Electrical and Electronics Engineering/ Electronics and Communication Engineering/ Electronics and Instrumentation Engineering/ Geoinformatics Engineering/ Industrial Engineering/ Industrial Engineering and Management/ Instrumentation and Control Engineering/ Manufacturing Engineering/ Marine Engineering/ Materials Science and Engineering/Mechanical Engineering/Mechanical and Automation Engineering/ Mechatronics Engineering/ Medical Electronics/ Petrochemical Engineering/ Production Engineering/ Robotics and Automation Engineering/ Biotechnology, Chemical Engineering/ Chemical and Electrochemical Engineering/ Food Technology/ Information Technology/ Petrochemical Technology/ Petroleum Engineering/ Plastic Technology/Polymer Technology)

(Regulations 2013)

Time : Three Hours

Maximum : 100 Marks

Answer ALL questions.

M (B2)

PART – A (10×2=20 Marks)

(Sn) 1. Find the partial differential equation by eliminating the arbitrary function 'f' from the relation $z = f(x^2 - y^2)$.

2. Find the complete integral of $\sqrt{p} + \sqrt{q} = 1$.

IT-1 (1)

3. State Dirichlet's conditions for a given function f(x) to be expanded in Fourier series.

4. Write the complex form of Fourier series for a function f(x) defined in $-l < x < l$.

IT-1 (6)

5. What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation ?

IT-3 (3)

6. State any two solutions of the Laplace equation $u_{xx} + u_{yy} = 0$ involving exponential terms in x or y.

IT-3 (10)

7. If $F[f(x)] = F(s)$, then find $F[f(ax)]$.

IT-2 (6)

8. State the convolution theorem for Fourier transforms.

IT-2 11-b

9. Find the Z-transform of the function $f(n) = 1/n$. Model-8 (SM)

10. Form the difference equation by eliminating arbitrary constant 'a' from $y_n = a.\,2^n$. Model-(9)

PART – B (5×16=80 Marks)

11. a) i) Find the singular integral of $z = px + qy + p^2 - q^2$. M-11-c (SM) (8)

ii) Find the general integral of $(x - 2z)\,p + (2z - y)\,q = y - x$. M-11-a (SM) (8)

(OR)

b) Solve the following equations.

i) $(D^2 + 2DD' + D'^2)\,z = e^{x-y} + xy$ M-15-d (SM) (8)

ii) $(D^2 - 5DD' + 6D'^2)\,z = y \sin x$. M-13-a (SM) (8)

12. a) i) Find the Fourier series for a function $f(x) = x + x^2$ in $(-\pi, \pi)$ and hence deduce the value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$ IAT-1-13-d. (SM) (8)

ii) Find the Fourier series of $y = f(x)$ up to first harmonic which is defined by the following data in $(0, 2\pi)$: IAT-1-11-b

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x)	1	1.4	1.9	1.7	1.5	1.2	1

(8)

(OR)

b) i) Find the half-range cosine series for $f(x) = x$ in $(0, \pi)$. Hence deduce the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$ IAT-1-13(c) (8)

ii) Find the Fourier series for a function $f(x) = \begin{cases} l - x, & 0 < x \le l \\ 0, & l < x \le 2l \end{cases}$ in $(0, 2l)$. (8)

13. a) A tightly stretched string of length l has its end fastened at $x = 0$, $x = l$. At $t = 0$, the string is in the form $f(x) = kx(l - x)$ and then released. Find the displacement at any point of the string at a distance x from one end and at any time $t > 0$. **(16)**

IAT-3 11-a

(OR)

b) A rod of length l cm has its ends A and B kept at 0°C and 100°C respectively, until steady state conditions prevail. If the temperature at B is suddenly reduced to 0°C and maintained at 0°C, find the temperature distribution $u(x, t)$ at a distance x from A at any time t. **(16)**

IAT-3 13 b (SM)

14. a) i) If $F_S(s)$ and $F_C(s)$ denote Fourier sine and cosine transform of a function $f(x)$ respectively, then show that

$$F_S\{f(x)\sin ax\} = \frac{1}{2}\{F_C(s-a) - F_C(s+a)\}$$ **(4)**

IT-2 8 (SM)

ii) Find the Fourier transform of a function $f(x) = \begin{cases} 1-|x| \text{ if } -1 < x < 1 \\ 0, \quad \text{otherwise} \end{cases}$ and hence

IT-2 11-a

find the value of $\int_0^\infty \frac{\sin^4 t}{t^4} dt$ by Parseval's identity. **(12)**

(OR)

b) Find the Fourier sine and cosine transforms of a function $f(x) = e^{-x}$. Using Parseval's identity, evaluate :

IAT-2 12-b (SM) 15-b

(1) $\int_0^\infty \frac{dx}{(x^2+1)^2}$ and (2) $\int_0^\infty \frac{x^2 dx}{(x^2+1)^2}$ **(16)**

15. a) i) Find the Z-transform of $\frac{2n+3}{(n+1)(n+2)}$. **(8)**

ii) Find $Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]$ by using convolution theorem. **(8)**

IAT-3 15(a)

(OR)

b) i) Find the inverse Z-transform of $\frac{z^3}{(z-1)^2(z-2)}$ by method of partial fraction. **(6)**

ii) Solve the difference equation $y(n+2) - 7y(n+1) + 12y(n) = 2^n$, given that $y(0) = 0$ and $y(1) = 0$, by using Z-transform. **(10)**

~~IAT~~ M - 12-d (SM)

Reg. No. : [ ][ ][ ][ ][ ][ ][ ][ ][ ][ ][ ][ ]

Question Paper Code : 41310

B.E./B.Tech. DEGREE EXAMINATION, APRIL/MAY 2018

Third Semester

Civil Engineering

MA 6351 – TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

(Common to Mechanical Engineering (Sandwich)/Aeronautical Engineering/ Agriculture Engineering/Automobile Engineering/Biomedical Engineering/ Computer Science and Engineering/Computer and Communication Engineering/ Electrical and Electronics Engineering/Electronics and Communication Engineering/Electronics and Instrumentation Engineering/Geoinformatics Engineering/Industrial Engineering/Industrial Engineering and Management/ Instrumentation and Control Engineering/Manufacturing Engineering/Marine Engineering/Materials Science and Engineering/Mechanical Engineering/ Mechanical and Automation Engineering/Mechatronics Engineering/Medical Electronics/Petrochemical Engineering/Production Engineering/Robotics and Automation Engineering/Bio Technology/Chemical Engineering/Chemical and Electrochemical Engineering/Food Technology/Information Technology/ Petrochemical Technology/Petroleum Engineering/Plastic Technology/ Polymer Technology)

(Regulations 2013)

Time : Three Hours | Maximum : 100 Marks

Answer ALL questions.

PART – A (10×2=20 Marks)

1. Find the complete integral of the PDE : $z = px + qy + \sqrt{pq}$

2. Solve : $(D^3 - 3DD'^2 + 2D'^3)\, z = 0$

3. Find b_n in the expansion of $f(x) = x^2$ as a Fourier series in $(-\pi, \pi)$.

4. Define Root mean square value of a function.

5. Classify the partial differential equation $u_{xy} = u_x u_y + xy$.

6. State possible solutions of the one dimensional heat equation.

7. Find the Fourier transform of $f(x)=\begin{cases}1 & \text{in} |x|<a \\ 0 & \text{in} |x|>a\end{cases}$

8. State the convolution theorem for Fourier transforms.

9. Find the Z-transform of {n}.

10. Prove that $Z\{nf(n)\} = -z\frac{d}{dz}F(z)$, where $Z\{f(n)\} = F(z)$.

PART – B

(5×16=80 Marks)

11. a) i) Find the singular solution of the equation $z = px + qy + p^2 + pq + q^2$. **(8)**

ii) Solve : $x(y - z)p + y(z - x)q = z(x - y)$. **(8)**

(OR)

b) i) Solve : $(D^2 + 4DD' - 5D'^2)z = \sin(x - 2y) + e^{2x-y}$. **(8)**

ii) Solve : $(D^2 + DD' - 6D'^2)z = y\cos x$. **(8)**

12. a) i) Find the Fourier series for $f(x) = x^2$ in $-\pi < x < \pi$. **(8)**

ii) Find the half range cosine series for $f(x) = x(\pi - x)$ in $(0, \pi)$. **(8)**

(OR)

b) Find the Fourier series expansion upto the first three harmonics for the function defined in the following table **(16)**

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1.0

13. a) A string is stretched and fastened to two points $x = 0$ and $x = l$ apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t. **(16)**

(OR)

b) A bar, 10 cm long with insulated sides, has its ends A and B kept at 20°C and 40°C respectively until steady state conditions prevail. The temperature at A is then suddenly raised to 50°C and at the same instant that at B is lowered to 10°C. Find the subsequent temperature at any point of the bar at any time. **(16)**

14. a) Find the Fourier transform of the function f(x) defined by $f(x)=\begin{cases}1-x^2 & \text{in } |x|\le 1\\ 0 & \text{in } |x|>1\end{cases}$.

Hence prove that $\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds=\frac{3\pi}{16}$. Also show that

$\int_0^{\infty}\frac{(x\cos x-\sin x)^2}{x^6}dx=\frac{\pi}{15}$. (16)

(OR)

b) i) Show that the Fourier transform of $f(x) = e^{\frac{-x^2}{2}}$ is $e^{\frac{-s^2}{2}}$. (8)

ii) Find the Fourier cosine transform of $e^{-a^2x^2}$ and hence find the Fourier sine transform of $xe^{-a^2x^2}$. (8)

15. a) i) Find the inverse Z-transform of $\frac{8z^2}{(2z-1)(4z+1)}$ using convolution theorem for Z-transforms. (8)

ii) Find the inverse Z-transform of $\frac{z^2-3z}{(z-5)(z+2)}$ using residue theorem. (8)

(OR)

b) i) Solve : $y_{n+2}-4y_{n+1}+4y_n=0$, $y_0=1$, $y_1=0$, using Z-transform. (10)

ii) Find the Z-transform of $\{n\}$ and $\left\{\frac{1}{n+1}\right\}$. (6)